Teichmüller's metric and Kobayashi's metric on the smooth Teichmüller space.

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Suppose $h:[0,1]\to[0,1]$ is a homeomorphism with $h(0)=0$, $h(1)=1$. Consider the quasisymmetric distortion,

$$\varepsilon(t) = \sup_{\substack{x\in[0,1] \\ x+t,\, x-t\in[0,1]}} \left| \log \left| \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \right| \right|$$

·) If $M = \sup_{t>0} \varepsilon(t) < \infty$, then we call h a M-quasisymmetric homeomorphism of $[0,1]$

··) If $\varepsilon(t) \to 0^+$ as $t\to 0^+$, then we call h a symmetric homeomorphism of $[0,1]$.

···) A C^1-diffeomorphism h is symmetric.

Let $\omega(t)$ be the modulus of continuity of $\log h'(x)$, that is, $\omega(t)>0$, increasing, $\omega(t)\to 0$ $t\to 0$ and $\omega(0)=0$, such that

$$\sup_{\substack{x,y\in[0,1] \\ |x-y|\le t}} \left| \log h'(x) - \log h'(y) \right| \le C\,\omega(t).$$

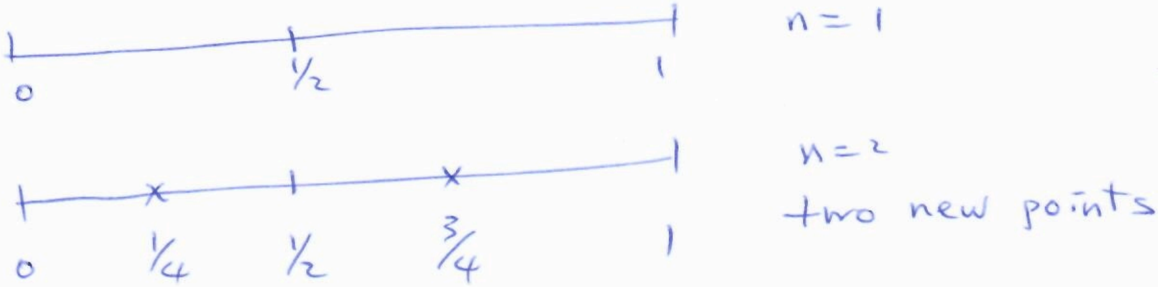
We say $h\in C^{1+\omega}$.

Furthermore, $\varepsilon(t) \le C\omega(t)$.

Let $I_{0,0} = [0,1]$

$$I_{n,k} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right], \qquad 0 \le k \le 2^n - 1$$

Then $I_{n-1,k} = I_{n,2k} \cup I_{n,2k+1}, \quad 0 \le k \le 2^{n-1} - 1$

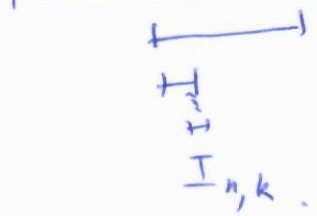


$$\Rightarrow \frac{1}{1+e^{\varepsilon(\frac{1}{2^n})}} |h(I_{n-1,k})| \le |h(I_{n,j})| \le \frac{1}{1+e^{\varepsilon(\frac{1}{2^n})}} |h(I_{n-1,k})|$$

$j = 2k$ or $2k+1$.

$$(*) \quad \prod_{i=m+1}^{n} \frac{2}{1+e^{\varepsilon(\frac{1}{2^i})}} \frac{|h(I_{m,l})|}{|I_{m,l}|} \le \frac{|h(I_{n,k})|}{|I_{n,k}|} \le \prod_{i=m+1}^{n} \frac{2}{1+e^{-\varepsilon(\frac{1}{2^i})}} \frac{|h(I_{m,l})|}{|I_{m,l}|}$$

for any $I_{n,k} \subset I_{m,l}$



Consider the infinite product

$$S = \prod_{i=1}^{\infty} \frac{1+e^{\varepsilon(\frac{1}{2^i})}}{2}$$

If S is convergent, then

$$\left\{ \log \frac{|h(I_{w_n})|}{|I_{w_n}|} \right\}_{n=1}^{\infty}$$

is a Cauchy sequence, $w_n \Leftrightarrow k$, i.e.

$k = i_0 2^{n-1} + i_1 2^{n-2} + \cdots + 2 i_{n-2} + i_{n-1}$, $\quad w = i_0 i_1 \cdots i_{n-1}$

$$\Rightarrow \quad \varphi(w) = \lim_{n\to\infty} \log \frac{|h(I_{w_n})|}{|I_{w_n}|} : \Sigma^+ \to \mathbb{R}^+$$

defines a continuous function, moreover,

$$\varphi(w_n 00\cdots) = \varphi(w_n 11\cdots)$$

$\Rightarrow$ φ pushdown to $[0,1]$ defines a continuous function on $[0,1]$.

S is convergent $\Longleftrightarrow$ $\tilde{\varepsilon}(t) = \int_0^{1/2} \frac{\varepsilon(t)}{t}\, dt < \infty$

$\uparrow$

$\varepsilon(t)$ is called Dini.

<u>Proposition 1</u>. If $\varepsilon(t)$ is Dini, then $h \in C^{1+\tilde{\varepsilon}(t)}$,

i.e. h is a diffeomorphism whose derivative has the modulus of continuity of $\tilde{\varepsilon}(t)$.

In particular, if $\varepsilon(t) = t^{\alpha}$, $0 < \alpha \le 1$, then $\tilde{\varepsilon}(t) = C t^{\alpha} \Rightarrow h$ is a $C^{1+\alpha}$ diffeomorphism.

In general $\varepsilon(t) \not\approx \tilde{\varepsilon}(t)$. Actually $\varepsilon(t) \approx \tilde{\varepsilon}(t) \iff \varepsilon(t) = t^{\alpha}$.

In general, a quasisymmetric homeomorphism (or symmetric homeomorphism) is not differentiable or even totally singular:

<u>Proposition 2</u> (The best estimated quasisymmetric distortion)

Suppose $h : [0,1] \to [0,1]$, $h(0) = 0$, $h(1) = 1$, is a M-quasisymmetric homeomorphism.

Then
$$|h(x) - x| \le M - 1, \quad \forall x \in [0,1]$$

The bound $M-1$ is the ~~best~~ sharpest estimation.

(see the paper for a detailed proof)

Now let us back to circle homeomorphism h (we will think h either on $\mathbb{T}$, or $\mathbb{R}$ or $[0,1]/_{0\sim 1}$)

A) Beurling-Ahlfor's extension.

Here we think h on $\mathbb{R}$.

$$H(x,y) = u + iv$$

$$u = \frac{1}{2}\int_0^1 \left(h(x+ty) + h(x-ty)\right)dt = \frac{1}{2y}\int_{x-y}^{x+y} h(t)\,dt$$

$$v = \frac{1}{y}\left(\int_x^{x+y} h(t)\,dt - \int_{x-y}^{x} h(t)\,dt\right)$$

$$\Rightarrow \quad \overline{H(z)} = H(\bar z), \quad H|_{\mathbb{R}} = h$$

$$|\mu_h(z)| = \left|\frac{H_{\bar z}}{H_z}(z)\right| \le C_0\left(e^{\varepsilon(y)} - 1\right)$$

$$\le C\,\varepsilon(y).$$

$\forall z = x + yi$

Proposition 3. Suppose $h \in C^{H\omega}$. Then

$$|\mu_h(z)| \leq C\omega(1-r) \quad 0<r<1, \; z \in \mathbb{C} \text{ near } \mathbb{T}$$

(Here we think h is on $\mathbb{T}$)

Let $BM(\Delta)$ = the unit ball of $L^\infty(\Delta)$

$$d_{BM}(\mu,\nu) = \tanh^{-1}\left\|\frac{\mu-\nu}{1-\bar{\mu}\nu}\right\|_\infty = \frac{1}{2}\log\left(\frac{1+\left\|\frac{\mu-\nu}{1-\bar{\mu}\nu}\right\|_\infty}{1-\left\|\frac{\mu-\nu}{1-\bar{\mu}\nu}\right\|_\infty}\right)$$

B) Projection $\mathcal{P}$:

$\forall \mu \in BM(\Delta)$, define

$$\tilde{\mu}(z) = \begin{cases} \mu(z), & z \in \Delta \\ \mu(z^*)^*, & z \in \Delta_\infty = \hat{\mathbb{C}} \setminus \Delta, \end{cases}$$

z^* is the reflection of z wrt $\mathbb{T}$.

Let w_μ be the quasiconformal mapping whose Beltrami coefficient is $\tilde{\mu}$. Then $w_\mu|\mathbb{T}$ is a circle quasisymmetric homeomorphism.

$$\mathcal{P}(\mu) = [w_\mu|\mathbb{T}] \in TQ : BM(\Delta) \to TQ$$

It is ~~??~~ onto and $\mathcal{P}(BM(\Delta)) = TQ$

let $A_r = \{ z \mid 1-r < |z| < 1 \}$, $0 < r < 1$.

For a modulus of continuity $\omega(t)$,

let $BM^{\omega}(\Delta) = \{ \mu \in BM(\Delta) \mid \| \mu | A_r \|_{\infty} \leq C\, \omega(1-r) \}$

<u>Proposition 4</u>. $\mathcal{P} : BM^{\omega}(\Delta) \to T\mathcal{C}^{1+\omega}$

is ~~not~~ onto and $\mathcal{P}(BM^{\omega}(\Delta)) \supset T\mathcal{C}^{1+\omega}$.

For a general $\mu \in BM^{\omega}(\Delta)$, we have only

$$w_{\mu}|\mathbb{T} \in C^{1+\hat{\omega}}.$$

for $\hat{\omega}(t) = t^{1-\beta} + \tilde{\omega}(t^{\beta})$

$$\tilde{\omega}(t^{\beta}) = \int_0^{1/2} \frac{\omega(t^{\beta})}{t}\, dt, \quad 0 < \beta < 1.$$

$\Rightarrow$ We may not have "$=$".

Our estimation re-verify that $\mathcal{P}(BM^{0}(\Delta)) = TS$

where $BM^{0}(\Delta) = \{ \mu \in BM(\Delta) \mid \| \mu | A_r \|_{\infty} \to 0, r \to 1 \}$

c) Bers embedding.

$\forall \mu \in BM(\Delta)$, let

$$\hat{\mu}(z) = \begin{cases} \mu(z), & z \in \Delta \\ 0 & z \in \Delta_\infty = \hat{\mathbb{C}} \setminus \Delta. \end{cases}$$

Let w^μ be the quasiconformal mapping whose Beltrami coefficient is $\hat{\mu}$

$\Rightarrow$ $w^\mu|\Delta_\infty$ is holomorphic

Let $S(w^\mu) = \left(N'(w^\mu) - \frac{1}{2}\left(N(w^\mu)\right)^2\right) dz^2$

be the Schwarzian derivative, where

$$N(w^\mu) = \frac{(w^\mu)''}{(w^\mu)'} = \cancel{\ }(Dw^\mu)' \leftarrow \text{nonlinearity}$$

$$Dw^\mu = \log (w^\mu)'. \leftarrow \text{derivative.}$$

Let QD = the space of all holomorphic quadratic differentials $q = \varphi dz^2$ on Δ_∞ with

$$\|q\| = \sup_{z \in \Delta_\infty} |\varphi(z) \rho_\infty^{-2}(z)|$$

where

$$\rho_\infty(z)|dz| = \frac{|dz|}{|z|^2 - 1}$$ is the hyperbolic metric on Δ_∞.

Then $\iota : \tau = [\mu] \in TQ \to S w^{\mu} \in QD$

is the Bers embedding such that

$$i(TQ) \subseteq B_6 \subset QD$$

open

B_6 is the ball of radius 6 centered at 0.

Let $QD^{\omega} = \{ \varphi \in QD \mid |\varphi(z) \rho_{\omega}^{-2}(z)| \leq C \omega(1 - \frac{1}{|z|}) \}$

for a modulus of continuity $\omega(t)$.

Then we have for $\tau = [h] \in T\mathcal{C}^{1+\omega}$,

$$|S w^{\mu}(z) \rho_{\omega}^{-2}(z)| \leq C \left(\left(1 - \frac{1}{|z|}\right)^{2(1-\beta)} + \omega\left(\left(1 - \frac{1}{|z|}\right)^{\beta}\right) \right)$$

$\frac{1}{2} < \beta < 1$

~~[illegible]~~.

$$\Rightarrow \iota(\tau) \in QD^{\tilde{\omega}}$$

$$\tilde{\omega} = t^{2(1-\beta)} + \omega(t^{\beta})$$

Consider B) and c) if we want

$\omega(t)$, $\hat{\omega}(t)$, $\tilde{\omega}(t)$ in the same class of

modulus of continuity, then $\omega(t) = t^{\alpha}$

($\Rightarrow$ $\hat{\omega}(t) = t^{\hat{\alpha}}$, $\tilde{\omega}(t) = t^{\tilde{\alpha}}$ are all Hölder)

consider

$$Te^{1+H} = \bigcup_{0<\alpha\leq 1} Te^{1+\alpha}$$

$$BM^{H}(\Delta) = \bigcup_{0<\alpha\leq 1} BM^{\alpha}(\Delta)$$

$$QD^{H} = \bigcup_{0<\alpha\leq 1} QD^{\alpha}(\Delta)$$

Then we have

$$\mathcal{P} : BM^{H}(\Delta) \to Te^{1+H} \xhookrightarrow{i} QH^{H}$$

and $\mathcal{P}(BM^{H}(\Delta)) = Te^{1+H}$

thus we have the Bers complex manifold structure on Te^{1+H} such that $\mathcal{P}$ is a holomorphic split submersion.

$$d_{BM}(\mu,\nu) = \tanh^{-1}\left(\left\|\frac{\mu-\nu}{1-\bar{\mu}\nu}\right\|_\infty\right)$$

induces the Teichmüller metric on TQ

$$d_T(\tau,\tau') = \inf\{ d_{BM}(\mu,\nu) \mid P(\mu)\in\tau, P(\nu)\in\tau' \}$$

<u>$d_{T,H}$</u> be the restriction of d_T on TC^{HH}.

Since TC^{HH} is a complex Banach manifold, it has a natural Kobayashi metric $d_{K,H}$ which is the largest pseudo metric on TC^{HH} such that

$$d_{K,H}(f(z), f(w)) \leq \rho_\Delta(z,w), \quad z,w\in\Delta$$

for any holomorphic map $f:\Delta\to TC^{HH}$.

where $\rho_\Delta(z,w) = \tanh^{-1}(z,w) = \frac{1}{2}\log\dfrac{1+\left|\frac{z-w}{1-\bar{z}w}\right|}{1-\left|\frac{z-w}{1-\bar{z}w}\right|}$

be the hyperbolic metric on Δ.

Then we have

<u>Lemma 1,</u> $\forall\ \tau, \tau' \in \mathcal{TC}^{HH}$,

$$d_{K,H}(\tau,\tau') \geq d_{T,H}(\tau,\tau')$$

Proof. Since $\mathcal{TC}^{HH} \subset \mathcal{TQ}$

$$\Rightarrow \quad d_{K,H}(\tau,\tau') \geq d_K(\tau,\tau') = d_T(\tau,\tau') = d_{T,H}(\tau,\tau')$$ ※

<u>Lemma 2.</u> $\forall\ \tau, \tau' \in \mathcal{TC}^{HH}$,

$$d_{K,H}(\tau,\tau') \leq d_{T,H}(\tau,\tau')$$

Lemma 1 and Lemma 2 $\Rightarrow$

Theorem: On $\mathcal{TC}^{HH}$, $d_{K,H} = d_{T,H}$.

We need to prove Lemma 2.

moves η to $\mathcal{P}(0) = [0]$ and preserves both Teichmüller's metric and Kobayashi's metric. Thus to prove (5.7) for any points $\tau, \eta \in \mathcal{TC}^{1+H}$, we only need to prove

$$d_{K,H}([0], \tau) \leq d_{T,H}([0], \tau). \tag{5.8}$$

Before to prove this inequality, we review some properties in Teichmüller theory without proofs. The reader who is interested in them may refer to [12, 19, 26].

5.1 Extremal Point

Suppose ϕ is a holomorphic function on Δ. Let

$$\|\phi\| = \int_{\Delta} |\phi(z)| dxdy, \quad z = x + \mathrm{i}y.$$

Given a point $\tau = [\mu] \in \mathcal{TQ}$, let

$$k_0 = \inf_{\mu \in \tau} \|\mu\|_{\infty}.$$

From the normal family theory in quasiconformal theory, we have a $\mu_0 \in \tau$ such that $\|\mu_0\|_{\infty} = k_0$. We call μ_0 an extremal point in τ.

A sequence $\{\varphi_n\}$ of holomorphic functions is called a *Hamilton sequence* for μ_0 if $\|\phi_n\| = 1$ and $\lim_{n\to\infty} \sup \int_{\Delta} \mu_0 \varphi_n dxdy = \|\mu_0\|_{\infty}$.

Theorem 5.4 (Hamilton–Krushkal Theorem) *Given any point $\tau = [\mu] \in \mathcal{TQ}$, if $\mu_0 \in \tau$ is an extremal point, then μ_0 has a Hamilton sequence $\{\phi_n\}$.*

5.2 Frame Point

Given a point $\tau = [\mu] \in \mathcal{TQ}$, an element $\mu_1 \in \tau$ is called a frame point if there is a compact set $D \subset \Delta$ such that

$$\|\mu_1 | (\Delta \setminus D)\|_{\infty} < k_0.$$

Lemma 2.9 says that if $\tau \neq [0] \in \mathcal{TC}^{1+H}$, then it always has a frame point.

Theorem 5.5 (Strebel's Frame Mapping Theorem) *For ant $\tau \neq [0] \in \mathcal{TQ}$, if it has a frame point, then it has a unique extremal point μ_0 in the Teichmüller form,*

$$\mu_0 = k_0 \frac{|\varphi_0|}{\varphi_0},$$

for a holomorphic function φ_0 with $\|\phi_0\| = 1$. Moreover, for any $\nu \in \tau$,

$$K_0 = \frac{1 + k_0}{1 - k_0} \leq \int_{\Delta} \frac{|1 + \nu \frac{\varphi_0}{|\varphi_0|}|^2}{1 - |\nu|^2} |\varphi_0| dxdy.$$

5.3 Holomorphic Functions

Suppose $\{\varphi_n\}$ is a sequence of holomorphic functions with $\|\phi_n\| = 1$. Suppose $D \subset \Delta$ is a compact subset. We claim that $\{\varphi_n\}$ is uniformly bounded on D. We prove the claim by contradiction. Suppose not, then there exists a sequence of points $\{z_n\} \subset D$ and a subsequence of $\{\varphi_n\}$, still denoted by $\{\varphi_n\}$, such that $|\varphi_n(z_n)| \geq n$. Since D is compact, $\{z_n\}$ has an accumulation point $z_0 \in D$. Then there exists a subsequence of $\{z_n\}$, still denoted by $\{z_n\}$, such that z_n converges to z_0. Choose a small $r > 0$ such that the closed disk $D_r(z_0) = \{|z - z_0| \leq r\} \subset \Delta$. Then $z_n \in D_{r/4}(z_0)$ when n is large enough, say $n > N$.

For any $n > N$, one can apply the Cauchy integral formula for $\varphi_n(z_n)$ to obtain

$$n \leq |\varphi_n(z_n)| \leq \frac{1}{2\pi} \int_{|z-z_0|=r'} \frac{|\varphi_n(z)|}{|z-z_n|} r' d\theta$$

for each $\frac{r}{2} \leq r' \leq r$. And then

$$n \leq \frac{1}{2\pi} \int_{|z-z_0|=r'} |\varphi_n(z)| \frac{4}{r} r d\theta = \frac{2}{\pi} \int_{|z-z_0|=r'} |\varphi_n(z)| d\theta.$$

Multiplying the previous inequality by r' and integrating both sides in radial direction from $\frac{r}{2}$ to r, we obtain

$$\frac{3}{8} n r^2 = n \int_{\frac{r}{2}}^{r} r' dr' \leq \frac{2}{\pi} \int_{\frac{r}{2}}^{r} r' \int_{|z-z_0|=r'} |\varphi_n(z)| d\theta dr' \leq \frac{2}{\pi} \|\varphi_n\| = \frac{2}{\pi}.$$

Hence $\frac{3}{8} n r^2 \leq \frac{2}{\pi}$ for any $n > N$. This is a contradiction when n is large enough. We proved the claim.

Applying the Cauchy integral formula for derivatives $\{\phi_n'\}$, one can see it is also uniformly bounded on D and thus $\{\phi_n\}$ is a uniformly bounded equi-continuous family. The Ascoli–Arzela Theorem implies $\{\phi_n\}$ has a convergent subsequence, still denoted as $\{\phi_n\}$, on D. Taking an increasing sequence of compact sets $\{D_m\}$ such that $\Delta = \bigcup_m D_m$, we get a convergent subsequence of $\{\phi_n\}$, still denoted as $\{\phi_n\}$, on Δ. Suppose ϕ_0 is its limiting function. By Fatou's Lemma, $\|\phi_0\| \leq 1$.

5.4 The Proof of Lemma 5.3

For any $\tau \in \mathcal{TC}^{1+H}$, take $\mu \in \tau$ in Lemma 2.9. Let $k = \|\mu\|_\infty$. Let

$$\Delta_n = \left\{ z \in \Delta \;\middle|\; |z| < r_n = 1 - \frac{1}{n} \right\} \quad \text{and} \quad A_n = \Delta \setminus \Delta_n.$$

Let $l_n = \|\mu|A_n\|_\infty$. Lemma 2.9 implies that $l_n < k_0$ for n large enough, say $n > N$. So μ is a frame point in τ. This implies that τ has a unique extremal point μ_0 in the Teichmüller form $\mu_0 = k_0 |\phi_0| / \phi_0$ for some holomorphic function ϕ_0 with $\|\phi_0\| = 1$. Moreover, $0 < k_0 < k$.

Let $f_n(z) = w_\mu(r_n z)$. It maps Δ to a quasi-disk $D_n = f_n(\Delta)$. Let $g_n : D_n \to \Delta$ be the Riemann mapping. Then $h_n = g_n \circ f_n$ is a quasiconformal self-homeomorphism of Δ and $\tau_n = [h_n|T]$ is in $\mathcal{TQ}$. From Lemma 2.9, for N large enough, every point τ_n has a frame point for $n > N$. Thus for every $n > N$, τ_n has a unique extremal point $\mu_{n,0}$ in the Teichmüller form,

$$\mu_{n,0} = k_{n,0} \frac{|\phi_{n,0}|}{\phi_{n,0}}$$

with a holomorphic function $\phi_{n,0}$ with $\|\phi_{n,0}\| = 1$. By our definition, one can see that $k_{n,0} \geq k_0$ for all $n > N$.

Now we define $F_n(z) = g_n^{-1} \circ w_{\mu_{n,0}}(z/r_n)$ for $z \in \Delta_n$ and $F_n(z) = w_\mu(z)$ for $z \in A_n$. It agrees on the circle $\partial \Delta_n$. Thus it is a quasiconformal self-homeomorphism of Δ. The Beltrami coefficient ν_n of F_n is $\mu_{n,0}(z/r_n)$ on Δ_n and μ on A_n. Thus $\nu_n \in \tau \in \mathcal{TC}^{1+\alpha}$. And $\|\nu_n\|_\infty > k_0$. We have a holomorphic map

$$p(c) = \left[c \frac{\nu_n}{\|\nu_n\|_\infty} \right] : \Delta \to \mathcal{TC}^{1+\alpha}$$

such that $p(0) = [0]$ and $p(\|\nu_n\|_\infty) = \tau$. This implies that

$$d_{K,\alpha}([0], \tau) \leq d_1([0], \tau) \leq \frac{1}{2} \log \frac{1 + \|\nu_n\|_\infty}{1 - \|\nu_n\|_\infty}.$$

Our final step is to prove $\|\nu_n\|_\infty \to k_0$ as $n \to \infty$.

From Subsection 5.3, there exists a subsequence of $\{\varphi_{n,0}\}$, still denoted by $\{\varphi_{n,0}\}$, converging uniformly to a holomorphic function $\widehat{\varphi}$ on any compact subset $D \subset \Delta$. Furthermore, $\|\widehat{\phi}\| \leq 1$. We claim that $\|\widehat{\phi}\| > 0$. We prove the claim by contradiction.

Suppose $\|\widehat{\varphi}\| = 0$. Then $\{\varphi_{n,0}\}$ has a subsequence, we still denote by $\{\phi_{n,0}\}$, converging uniformly to zero on any compact subset $D \subset \Delta$. For any $\epsilon > 0$, we first choose a compact subset $D \subset \Delta$ such that

$$\|\mu|(\Delta \setminus D)\|_\infty < \epsilon.$$

There exists $N_1 > N$ such that

$$\int_D |\varphi_{n,0}(z)| dxdy \leq \epsilon$$

and such that $D \subset \Delta_n$ for all $n > N_1$.

From Subsection 5.2,

$$K_{n,0} = \frac{1 + k_{n,0}}{1 - k_{n,0}} \leq \int_\Delta \frac{|1 + \mu \frac{\varphi_{n,0}}{|\varphi_{n,0}|}|^2}{1 - |\mu|^2} |\varphi_{n,0}| dxdy.$$

This says

$$K_{n,0} \leq \int_{\Delta \setminus D} \frac{|1 + \mu \frac{\varphi_{n,0}}{|\varphi_{n,0}|}|^2}{1 - |\mu|^2} |\varphi_{n,0}| dxdy + \int_D \frac{|1 + \mu \frac{\varphi_{n,0}}{|\varphi_{n,0}|}|^2}{1 - |\mu|^2} |\varphi_{n,0}| dxdy.$$

Then, for $K = (1 + k)/(1 - k)$,

$$K_{n,0} \leq \int_{\Delta \setminus D} \frac{1 + \epsilon}{1 - \epsilon} |\varphi_{n,0}| dxdy + K \int \int_D |\varphi_{n,0}| dxdy,$$

and hence

$$K_{n,0} \leq \frac{1 + \epsilon}{1 - \epsilon} \int_\Delta |\varphi_{n,0}| dxdy + \left(K - \frac{1 + \epsilon}{1 - \epsilon}\right) \int \int_D |\varphi_{n,0}| dxdy.$$

Therefore

$$1 < k_0 < k_{n,0} \leq \frac{1 + \epsilon}{1 - \epsilon} + \left(K - \frac{1 + \epsilon}{1 - \epsilon}\right) \cdot \epsilon.$$

This is a contradiction when ϵ is sufficient small. Therefore $\|\widehat{\varphi}\| > 0$.

Now let $\widehat{\mu} = \widehat{k} \frac{|\widehat{\varphi}|}{\widehat{\varphi}}$, where $\widehat{k} = \lim_{n \to \infty} k_{n,0}$ (by taking a limit of a convergent subsequence if it is necessary). Then $\mu_{n,0} \to \widehat{\mu}$ a.e. on Δ. By the convergence theorem (see [20, Theorem 4.6]) of families of quasiconformal maps, we obtain

$$\lim_{n \to \infty} w_{\mu_{n,0}} | T = w_{\mu_{n,0}} | T = w_\mu | T = w_{\widehat{\mu}} | T.$$

By the uniqueness of the extremal point in τ, $\widehat{k} = k_0$. Thus $k_{n,0} \to k_0$ as $n \to \infty$ for a subsequence of n's. We completed the proof of Lemma 5.3. Both Lemmas 5.2 and 5.3 give a proof of Theorem 5.1.