Full Extendibility of Holomorphic Motions

Yunping Jiang

City University of New York–Queens College and Graduate Center

A talk given
in
The Summer Seminar on Quasiconformal Mappings and Teichmüller Spaces
in
Nanjing University of Science and Technology

August 13, 2021, 9:30am - 10:30am

Acknowledgment

This talk is based on the work I collaborated with Michael Beck and Zhe Wang (both are my my former students) and Fred Gardiner and Sudeb Mitra and Hiroshige Shiga.

Holomorphic Motions

Suppose V is a connected complex manifold with a basepoint t_0 and E is a subset of the Riemann sphere $\widehat{\mathbb{C}}$. A map

$$h(t,z) : V \times E \to \widehat{\mathbb{C}}$$

is called a holomorphic motion of E over V if

i) $h(t_0, z) = z$ for all $z \in E$;

ii) for any fixed $t \in V$, $h_t(\cdot) = h(t, \cdot) : E \to \widehat{\mathbb{C}}$ is injective;

iii) for any fixed $z \in E$, $h^z(\cdot) = h(\cdot, z) : V \to \widehat{\mathbb{C}}$ is holomorphic.

λ-Lemma

- $h(t,z)$ of E over V can be extended to a holomorphic motion $\overline{h}(t,z)$ of $\overline{E}$ over V.
- $\overline{h}(t,z) : V \times \overline{E} \to \widehat{\mathbb{C}}$ is a continuous map.
- $\overline{h}_t(\cdot) = \overline{h}(t,\cdot) : \overline{E} \to \widehat{\mathbb{C}}$ has some "quasiconformal property" for any given $t \in V$.

When $V = \Delta$, the open unit disk in the complex plane with 0 as the basepoint, this λ-lemma was first studied by Mañe-Sad-Sullivan and Lyubich in 1980s. For a general V, the proof is similar

Normalized Holomorphic Motions

Without loss of generality, we always assume that E is closed and contains $\{0, 1, \infty\}$ and that $h(t, 0) = 0$, $h(t, 1) = 1$, and $h(t, \infty) = \infty$ for all $t \in V$. In the rest of this talk, all holomorphic motions are assumed to be normalized.

The Extension Problem

Given a holomorphic motion h of E over V, can we extend it to a holomorphic motion H of the Riemann sphere $\widehat{\mathbb{C}}$ over V, that is, do we have a holomorphic motion

$$H(t,z) : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

such that $H|V \times E = h$?

If the answer is yes, then we call h fully extendable.

Simply Connected One-Dimensional Case

Theorem (Slodkowski's Theorem)

Any holomorphic motion $h(t, z) : \Delta \times E \to \widehat{\mathbb{C}}$ of E over Δ is fully extendable.

Slodkowski proved this theorem by using several complex variables in 1991. Chirka reproved it by using generalized Beltrami equations in partial differential equations in 2004 (as expected in Bers and Royden's paper, 1986). Several other proofs are also available from other people. One can read this in our survey article:

- Gardiner-J-Wang, 2010, Holomorphic motions and related topics. Geometry of Riemann Surfaces, London Mathematical Society Lecture Note Series, No. 368, 2010, 166-193.

Higher-Dimensional Simply Connected Case

There is a counter-example of a holomorphic motion h of E over a simply connected higher-dimensional complex manifold V such that h is NOT fully extendable, see, for example,

- J-Mitra, Some applications of universal holomorphic motions. Kodai Math J., Vol. 30, No. 1, 85-96).

Non-Simply Connected One-Dimensional Case, Example 1

Let $E = \{0, 1, \infty, t_0\}$ be a four-point set and $\mathbb{C}_{0,1} = \mathbb{C} \setminus \{0, 1\}$ be the thrice-punctured sphere with a basepoint t_0. The map

$$h_1(t, z) = \begin{cases} z & \text{if } z = 0, 1, \infty \text{ and } t \in \mathbb{C}_{0,1}; \\ t & \text{if } z = t_0 \text{ and } t \in \mathbb{C}_{0,1}. \end{cases}$$

is a holomorphic motion of E over $\mathbb{C}_{0,1}$. It is NOT full extendable.

This counter-example was first given by Douady and later Earle modified this example to get a counter-example of h_2 of a four-point subset $E = \{0, 1, \infty, t_0\}$ over an annulus A with a basepoint t_0, see

- Earle, 1997, Some maximal holomorphic motions, Contemp. Math., 211, 183-192.

A maximal property is used to show that h_1 (or h_2) is NOT fully extendable.

Non-Simply Connected One-Dimensional Case, Example 2

Let $E = \{0, 1, \infty, a = -t_0 + 2i, b = t_0 + 2i\}$ be a five-point set and $\Delta^* = \Delta \setminus \{0\}$ be the punctured unit disk with a basepoint t_0. The map

$$h_3(t, z) = \begin{cases} z & \text{if } z = 0, 1, \infty \text{ and } t \in \Delta^*; \\ -t + 2i & \text{if } z = a \text{ and } t \in \Delta^*; \\ t + 2i & \text{if } z = b \text{ and } t \in \Delta^*. \end{cases}$$

is a holomorphic motion of E over Δ^*. It is NOT fully extendable.

This example leads us to define monodromy for a general holomorphic motion in the paper

- Beck-J-Mitra-Shiga, 2012, Extending holomorphic motions and monodromy. Annales Academiæ Scientiarum Fennicæ Mathematica, Vol. 37 (2012), 53-67.

Winding Number

When Chirka gave a new proof of Slodkoswki's theorem in his paper 2004, he also asserted that the zero winding number condition, which is necessary, is also sufficient. However, it is NOT the case as we will see in the next slide.

Suppose $h : X \times E \to \widehat{\mathbb{C}}$ is a holomorphic motion of E over a hyperbolic Riemann surface X with a basepoint. Suppose $\alpha(\theta) : [0, 1] \to X$ is a simple closed curve. For any pair $z_1 \neq z_2 \in E$, let

$$\eta(\alpha, z_1, z_2) = \frac{1}{2\pi} \oint_{\alpha} d \arg \delta(\cdot, z_1, z_2)$$

be the winding number of the closed curve $\delta(\alpha, z_1, z_2) = h(\alpha, z_1) - h(\alpha, z_2)$ in $\mathbb{C}$ with respective to 0.

The Zero Winding Number Condition

We say that h satisfies the zero winding number condition if $\eta(\alpha, z_1, z_2) = 0$ for all α and all pairs $z_1 \neq z_2 \in E$.

All holomorphic motions in Example 1 and Example 2 do NOT satisfy the zero winding number condition. Therefore they are NOT fully extendable.

Non-Simply Connected One-Dimensional Case, Example 3

Let $E = \{0, 1, 2, 4, \infty\} \subset \widehat{\mathbb{C}}$ be a five-point set. Let A be a certain annulus such that $-2, 0, 1/2, 1/3, i, -i \notin A$. The map

$$\phi(t,z) = \begin{cases} z & \text{if } z = 0, 1, \infty \text{ and } t \in A; \\ -\frac{1}{t} + 3 & \text{if } z = 2 \text{ and } t \in A; \\ t + 3 & \text{if } z = 4 \text{ and } t \in A. \end{cases} \tag{1}$$

is a holomorphic motion of E over A such that it satisfies the zero winding number condition but is still NOT fully extendable. See details in my paper,

- J, 2020, Winding numbers and full extendibility in holomorphic motions. Conformal Geometry and Dynamics, AMS, May 26, 2020, Vol. 24, 109-117.

Characterization Problem

Since the zero winding number condition is NOT sufficient for the full extension problem, it is a good problem to find another sufficient (as well as necessary) topological condition for the full extension problem.

In the following slides, I will define the monodromy for a holomorphic motion and prove that the trivial monodromy condition is necessary and sufficient.

Measurable Riemann Mapping Theorem

Let

$$M(\mathbb{C}) = \{\mu \in L^{\infty}(\mathbb{C}) \mid \|\mu\|_{\infty} < 1\}$$

For any $\mu \in M(\mathbb{C})$, the Beltrami equation $w_{\overline{z}} = \mu w_z$ always has a solution w which is a K-quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ for $K = (1 + \|\mu\|_{\infty})/(1 - \|\mu\|_{\infty})$. Moreover, if we consider the normalized solution w^{μ} fixing 0, 1, ∞, then w^{μ} is unique and depends on μ holomorphically.

Teichmüller Space of a Closed Subset

Let E be a closed subset of $\widehat{\mathbb{C}}$. Suppose $0, 1, \infty \in E$. We say $\mu, \nu \in M(\mathbb{C})$ are E-equivalent if $(w^\nu)^{-1} \circ w^\mu$ is isotopic to the identity rel E. The space of all E-equivalence classes

$$T(E) = \{[\mu]_E \mid \mu \in M(\mathbb{C})\}$$

is called the Teichmüller space of the closed subset E. It is a simply connected complex Banach manifold such that the projection

$$P_E(\mu) = [\mu]_E : M(\mathbb{C}) \to T(E)$$

is a holomorphic split submersion.

Let

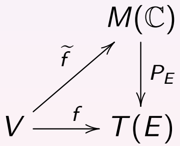
$$\Psi_E(t,z) = w^{\mu}(z) : T(E) \times E \to \widehat{\mathbb{C}}, \quad t = P_E(\mu),\ \mu \in M(\mathbb{C}).$$

It is a holomorphic motion of E over $T(E)$ with the basepoint $[0]_E$ and universal in the sense that for any holomorphic motion $h(t,z) : V \times E \to \widehat{\mathbb{C}}$ of E over simply connected complex Banach manifold V with a basepoint, we have a unique basepoint preserving holomorphic map $f : V \to T(E)$ such that

$$h(t,z) = f^*(\Psi_E)(t,z) := \Psi(f(t),z).$$

Lifting Problem

For a connected complex Banach manifold V with a basepoint and a given basepoint preserving holomorphic map $f : V \to T(E)$, when can we find a basepoint preserving holomorphic map $\widetilde{f} : V \to M(\mathbb{C})$ such that $P_E \circ \widetilde{f} = f$?



Lifting in the Simply Connected One-Dimensional Case

Theorem

Any basepoint preserving holomorphic map $f : \Delta \to T(E)$ can be lift to a basepoint preserving holomorphic map $\widetilde{f} : \Delta \to M(\mathbb{C})$ such that $P_E \circ \widetilde{f} = f$.

See our paper for the detailed proof:

- J-Mitra-Wang, 2009, Liftings of holomorphic maps into Teichmüller spaces, Kodai Mathematical Journal, 32 (2009), No. 3, 544-560.

This implies Slodkowski's theorem through the universal property of $\Psi_E(t, z)$. Actually, this is equivalent to the full extendibility.

The lifting theorem gives a simple proof of that $d_K = d_T$ on any Teichmüller space $T(E)$ (as well as on the Teichmüller space of a Riemann surface): From the definition, we have $d_K \leq d_T$, we only need to prove $d_K \geq d_T$. For any holomorphic map $f : \Delta \to T(E)$ such that $f(0) = 0$ and $f(c) = [\mu]$, we have a holomorphic map $\widetilde{f} : \Delta \to M$ such that $P_E \circ \widetilde{f} = f$. This implies that

$$d_T([0], [\mu]) \leq d_{K,M}(0, \widetilde{f}(c)) \leq d_{K,\Delta}(0, c).$$

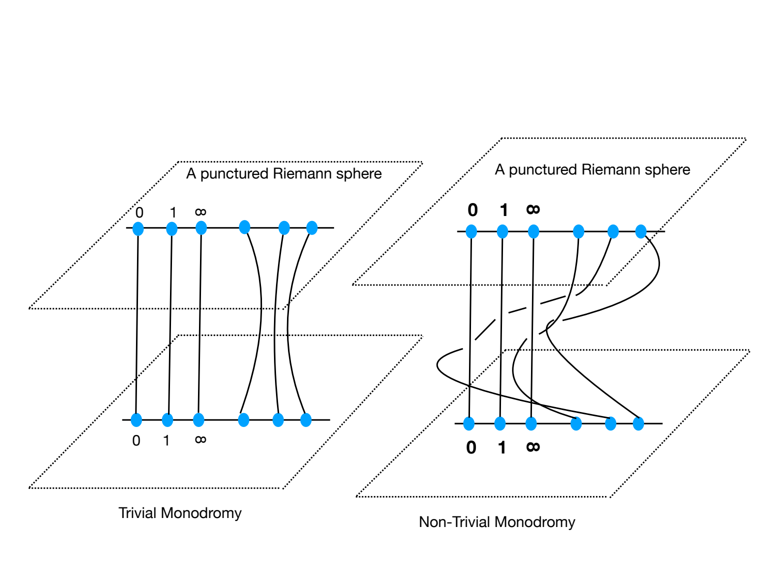
Take the infimum over all such f, we have

$$d_T([0], [\mu]) \leq d_K([0], [\mu]).$$

Monodomy for A Holomorphic Motion

Suppose X is a hyperbolic Riemann surface with a basepoint t_0 with the holomorphic universal cover $\pi : \Delta \to X$ with $\pi(0) = t_0$. Suppose $\{0, 1, \infty\} \subset E \subset \widehat{\mathbb{C}}$ is a finite subset. Given a holomorphic motion $h(t, z) : X \times E \to \widehat{\mathbb{C}}$, we have the pullback holomorphic motion $H(c, z) = \pi^*(h)(c, z) : \Delta \times E \to \widehat{\mathbb{C}}$, which gives a basepoint preserving holomorphic map $f : \Delta \to T(E)$ such that $H = f^*(\Psi_E)$. For any $\gamma \in \pi_1(X, t_0)$, let β be the representation of γ in the group Γ of deck transformations. Then the map w^μ for $P_E(\mu) = (f \circ \beta)(0)$ fixes every point in E. Thus it is a quasiconformal self-map of $X_E := \widehat{\mathbb{C}} \setminus E$ and represents a mapping class $[w^\mu]$ of X_E. The map $\rho_h(\gamma) = [w^\mu]$ from the fundamental group of X to the mapping group of X_E is called the monodromy for h.

Trivial and Non-Trivial Monodromy



Monodromy is Necessary

Theorem

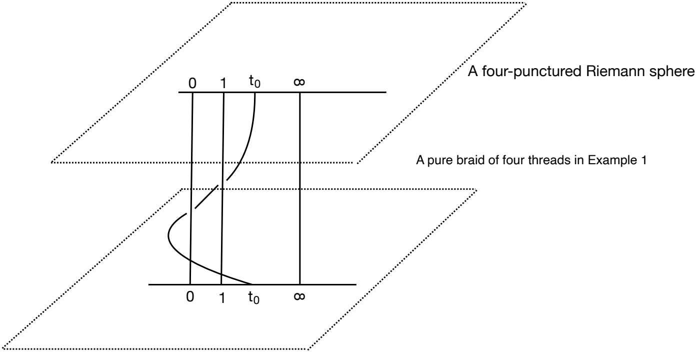
If a holomorphic motion $h : X \times E \to \widehat{\mathbb{C}}$ is fully extendable, then for any finite subset $\{0, 1, \infty\} \subset E' \subseteq E$, the monodromy $\rho_{E'}$ for $h' = h|X \times E'$ is trivial.

See details in the paper

- Beck-J-Mitra-Shiga, 2012, Extending holomorphic motions and monodromy. Annales Academiæ Scientiarum Fennicæ Mathematica, Vol. 37, 53-67.

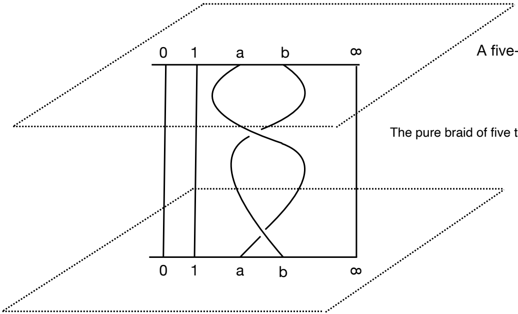
It is true when X is replaced by any connected complex Banach manifold V with a base.

Monodromy for Example 1



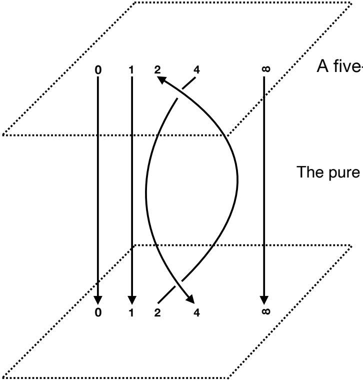
It has non-trivial monodromy, thus it is NOT fully extendable

Monodromy for Example 2



It has non-trivial monodromy, thus it is NOT fully extendable.

Monodromy for Example 3



A five-punctured Riemann sphere

The pure braid of five threads given in Example 3

It has non-trivial monodromy, thus it is NOT fully extendable

Monodromy is Sufficient

Theorem

Suppose $h(t,z) : X \times E \to \widehat{\mathbb{C}}$ is a holomorphic motion of E over a hyperbolic Riemann surface X with a basepoint t_0. If for any finite subset $\{0, 1, \infty\} \subset E' \subset E$, the monodromy $\rho_{E'}$ for $h' = h|X \times E'$ is trivial, then any holomorphic map $f : X \to T(E)$ can be lift to a holomorphic map $\widetilde{f} : X \to M(\mathbb{C})$ such that $f = P_E \circ \widetilde{f}$, and, moreover, h is fully extendable.

See details in the paper:

- J-Mitra, 2018, Monodromy, liftings of holomorphic maps, and extensions of holomorphic motions. Conformal Geometry and Dynamics, Volume 22 (2018), 333–344.

The Final Conclusion

Thus we completed this research program for the one-dimensional case by the following conclusion: The zero winding number condition and the trivial monodromy conditionare are all necessary for a fully extendable holomorphic motion. However, only the trivial monodromy condition is, indeed, sufficient.

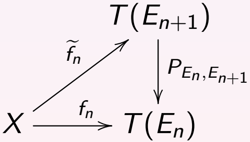
We used to study other two conditions, the trivial trace-monodromy condition and the guiding isotopy condition. Both are also necessary and the first one is not sufficient and the second one is sufficient. Now a more interesting research problem for the full extendibility of holomorphic motions is for the higher-dimensional case.

End

Thanks!

Proof of the Lifting Theorem, I

Suppose $E_n = \{0, 1, \infty, p_1, \cdots, p_n\}$ is a set of $n+3$ points and $E_{n+1} = E_n \cup \{p_{n+1}\}$ for $p_{n+1} \not\in E_n$. The key point in the whole proof is to find a holomorphic map $\widetilde{f_n} : X \to T(E_{n+1})$ such that $P_{E_n, E_{n+1}} \circ \widetilde{f_n} = f_n$, where f_n is the the holomorphic map constructed in the previous slide for $E_n = E'$ and $P_{E_n, E_{n+1}} : T(E_{n+1}) \to T(E_n)$ is the forgetful map.



Proof of the Lifting Theorem, II

Let

$$Y_n = \{\mathbf{c}_n = (z_1, \cdots, z_n) \mid z_i \neq 0, 1 \in \mathbb{C}, z_i \neq z_j, 0 \leq i < j \leq n\}$$

and, similarly, $Y_{n+1} = \{\mathbf{c}_{n+1}\}$. Then we have a natural projection $P(\mathbf{c}_{n+1}) = \mathbf{c}_n$. The maps

$$\pi_n([\mu]_{E_n}) = (w^\mu(p_1), \cdots, w^\mu(p_n)) : T(E_n) \to Y_n$$

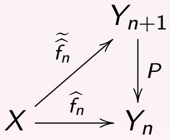
and, similarly, $\pi_{n+1}([\mu]_{E_{n+1}}) : T(E_{n+1}) \to Y_{n+1}$ are holomorphic universal covers, see Nag (1981, The Torelli spaces of punctured tori and spheres. Duke Math. J. 48, 359-388).

Proof of the Lifting Theorem, III

The map

$$\widehat{f}_n(t) = \pi_n \circ f_n(t) = (w^\mu(p_1), \cdots, w^\mu(p_n)) : X \to Y_n, \ \ P_{E_n}(\mu) = f_n(t)$$

is holomorphic. Now finding a lifting map $\widetilde{f}_n$ for f_n is equivalent to find $\widetilde{\widehat{f}}_n : X \to Y_{n+1}$ such that $P \circ \widetilde{\widehat{f}}_n = \widehat{f}_n$ and it is equivalent finding the last component $g_{n+1}(t) : X \to \mathbb{C}$ of $\widetilde{\widehat{f}}_n(t)$.



Proof of the Lifting Theorem, IV

For Δ, in J-Mitra-Wang (2009, Liftings of holomorphic maps into Teichmüller spaces, Kodai Mathematical Journal, 32 (2009), No. 3, 544-560), we have constructed a bounded operator

$$\mathcal{P} = \mathcal{P}(w^\mu(p_1), \cdots, w^\mu(p_n)) : C(\widehat{\mathbb{C}}) \to C(\widehat{\mathbb{C}})$$

such that it has a unique fixed point, that is,

$$g_{n+1} = \mathcal{P}(g_{n+1}).$$

Thus, for a general hyperbolic surface X, we can find $g_{n+1,0} : \Delta \to Y_{n+1}$ for $f_{n,0} = \widehat{f_n} \circ \pi : \Delta \to Y_n$.

Now we just need to check if $g_{n+1,0}$ is invariant under Γ. For any $\gamma \in \Gamma$, we have also

$$g_{n+1,0} \circ \gamma = (\mathcal{P} \circ \gamma)(g_{n+1,0} \circ \gamma)$$

where $\mathcal{P} \circ \gamma = \mathcal{P}(w^{\mu} \circ \gamma(p_1), \cdots, w^{\mu} \circ \gamma(p_n))$. When h satisfies the trivial monodromy condition, we have $\mathcal{P} \circ \gamma = \mathcal{P}$. Thus we have that

$$g_{n+1,0} \circ \gamma = \mathcal{P}(g_{n+1,0} \circ \gamma).$$

Since the fixed point of $\mathcal{P}$ is unique, $g_{n+1,0} = g_{n+1,0} \circ \gamma$. Thus, we find the last component

$$g_{n+1} = g_{n+1,0}/\Gamma : X \to \mathbb{C}$$

for $\widetilde{\widehat{f}}_n$.