part 1: symmetric circle homeomorphism and its extensions.

$\boxed{\text{Def 1}}$ A circle homeomorphism h is called quasisymmetric if there exists a constant $M \geq 1$ such that

$$\frac{1}{M} \leq \frac{H(x+t)-H(x)}{H(x)-H(x-t)} \leq M, \quad \forall x \in \mathbb{R}, \forall t > 0.$$

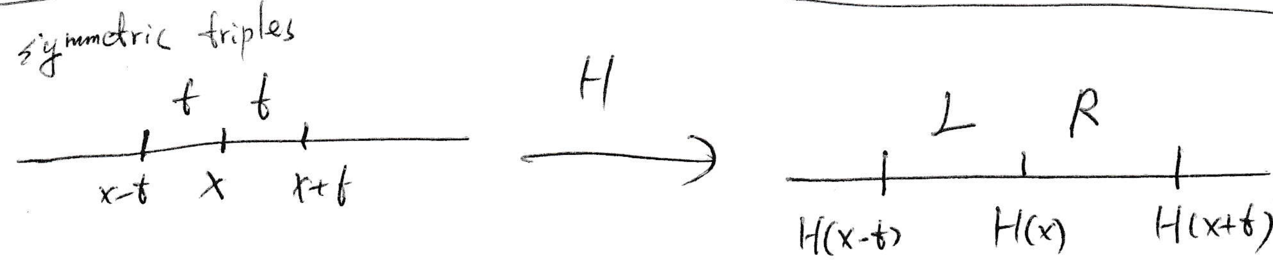
It is called symmetric, if there exists a positive function $\varepsilon(t)$ such that $\varepsilon(t) \to 0$ at $t \to 0^+$ and

$$\frac{1}{1+\varepsilon(t)} \leq \frac{H(x+t)-H(x)}{H(x)-H(x-t)} \leq 1+\varepsilon(t), \quad \forall x \in \mathbb{R}, \forall t > 0$$

Note: $h: S^1 \to S^1$ and $H: \mathbb{R} \to \mathbb{R}$

$\mathbb{R}$ is a universal cover of S^1: $\pi(x) = e^{2\pi i x}: \mathbb{R} \to S^1$

In this way, we can think $[0,1]$ as the unit circle S^1.



example of symmetric:
H is differentiable

QS: $\frac{1}{M} \leq \frac{R}{L} \leq M$

S: $\frac{1}{1+\varepsilon(t)} \leq \frac{R}{L} \leq 1+\varepsilon(t)$

Let's use unit disk model: Δ

$M(\Delta)$ = unit ball of the complex Banach space $L^\infty(\Delta)$

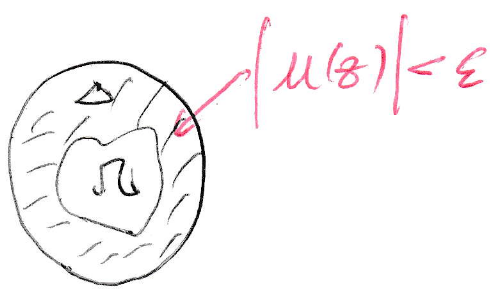
An element $\mu \in M$ is called a Beltrami coefficient on Δ.

Let f be a homeomorphism from Δ onto Δ satisfying

$$\frac{f_{\bar z}}{f_z} = \mu(z) \quad \text{for almost all } z \in \Delta.$$

Then f is quasiconformal. $(\|\mu(z)\|_\infty < 1)$

Def 2: A quasiconformal map f is said to be asymptotically conformal if for any $\varepsilon > 0$, there exists a compact subset Ω in Δ, such that $\|\mu\|_\infty\big|_{\Delta \setminus \Omega}$ is less than ε.



another understanding $\mu(z) \to 0$ as $z \to S^1$

Prop 1: suppose $h: S^1 \to S^1$ is the boundary map of $f: \Delta \to \Delta$

Then ① h is quasisymmetric $\Leftrightarrow$ f is quasiconformal.

② h is symmetric $\Leftrightarrow$ f is asymptotically conformal.

idea of proof of prop1:

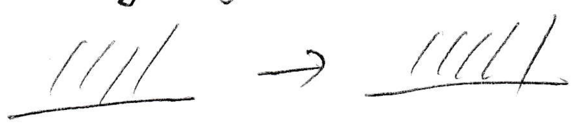
$\Rightarrow$ If h is quasisymmetric or symmetric, we can use Beurling-Ahlfors extension of h to calculate $\mu(z)$.

note: $h: \mathbb{R} \to \mathbb{R}$

$$BA(h)(x,y) = \cancel{}\, u(x,y) + V(x,y)\cdot i$$

where $u(x,y) = \frac{1}{2y}\int_{-y}^{y} h(x+t)\,dt$

$$V(x,y) = \frac{1}{y}\int_0^y \big(h(x+t) - h(x-t)\big)\,dt$$

BA(h): 

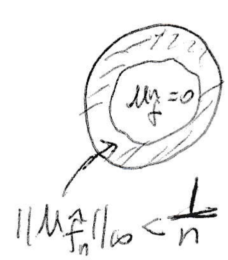
$\Rightarrow$ partial derivatives: u_x, u_y, V_x, V_y ...

$\Leftarrow$ We need a lemma for this direction.

lemma 1: If h is M-qs and ~~a [illegible]~~ a qc extention f is K-qc, then $M \le C(K)$ where $C(K) \to 1$, as $K \to 1$.

If f is aysmptotically conformal,

Define $\mu^n_{f_n} = \begin{cases} \mu_f & \text{on } \Delta / \Omega_n \\ 0 & \text{on } \Omega_n \end{cases}$



$$C(K^n_{f_n}) \to 1$$

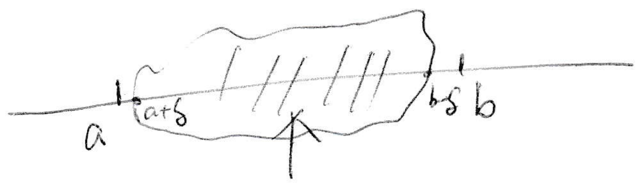
$$\Rightarrow M \to 1$$

note: 

$\mu_{f_n^*} = \begin{cases} 0 & \text{on } \Delta / \Omega_n \\ \mu_f & \text{on } \Omega_n \end{cases}$

Boundary map of $\mu_{f_n^*}$ is real analytic.

$\boxed{\text{prop 2}}$ suppose h is ~~symm~~ symmetric on an interval $[a,b]$, then $\exists\delta$ such that f is AC in a nbhd of $[a+\delta, b-\delta]$.



f is AC.

Note: ① Schwartzian derivative $\to 0$ as $|z|\to 1$.

② Douady-Earle extension: $\mu_{DE}(h) \to 0$ as $|z|\to 1$ for symmetric h. and it has similar result as prop 2.

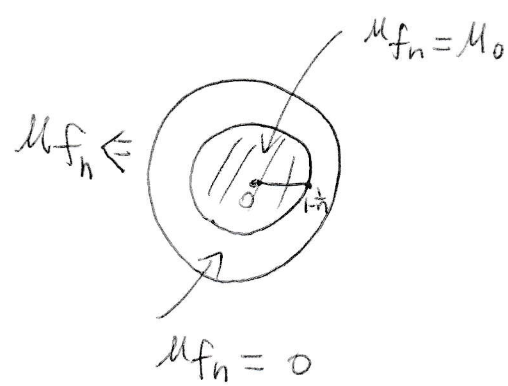
$\boxed{\text{Prop 3:}}$ Teichmuller's metric coinsides with Kobayashi's metric on T_0 = space of all symmetic h.

idea: By Strebel's frame mapping theorem,

$\exists\ \mu_0 \sim \mu_f$ and $\mu_0 = k_0 \frac{|\varphi_0|}{\varphi_0}$, Teichmuller form.

$[t\mu_0] \overset{?}{\in} T_0$

construct

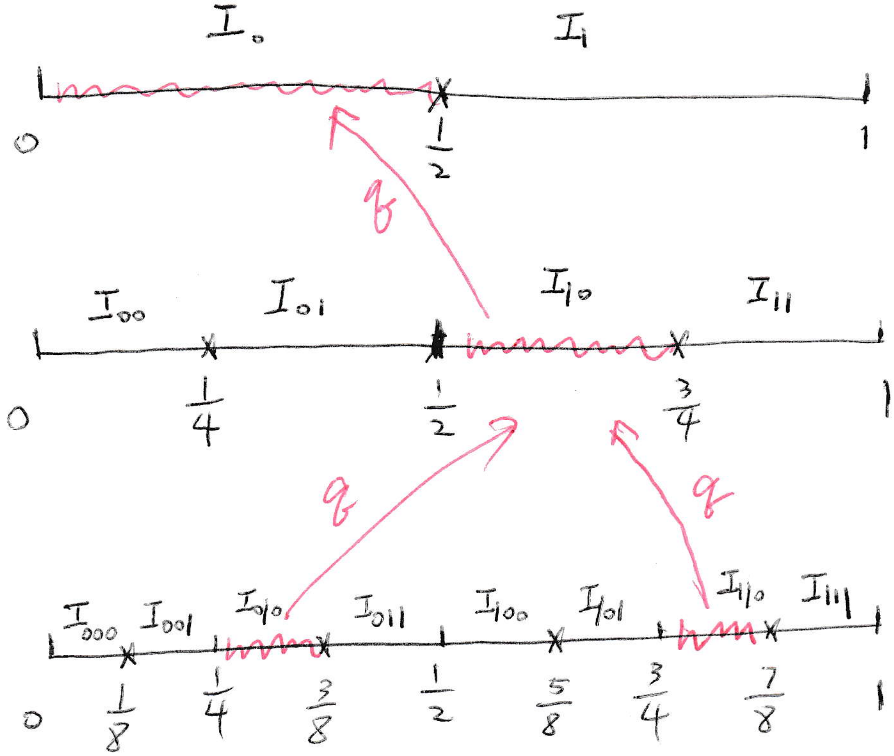


$[t\mu_{f_n}] \in T_0$.

part 2: Markov partition with Bounded Geometry

$q(z) = z^2$ on Δ or $q(x) = \begin{cases} 2x & x \in [0, \frac{1}{2}] \\ 2x-1 & x \in (\frac{1}{2}, 1] \end{cases}$

pre-images of 1: $q^{-n}(1)$



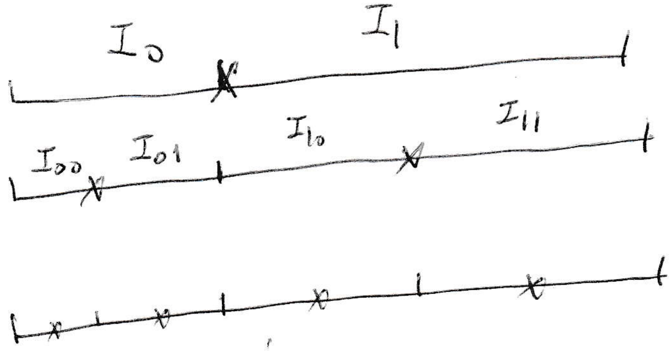
$\vdots$

I_{w_n} has two subintervals $I_{w_n 0}$ and $I_{w_n 1}$

I_{w_n} has two preimages $I_{0 w_n}$ and $I_{1 w_n}$

suppose f is a degree 2 circle endomorphism.

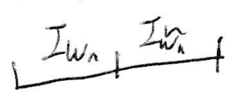
$f^{-n}(1)$ gives us a Markov partition.



⋮

① f has bounded geometry iff $\left|\frac{I_{w_n}}{I_{w_n 0}}\right| \leq M$, $\left|\frac{I_{w_n}}{I_{w_n 1}}\right| \leq M$ for any w_n

② f has bounded nearby geometry iff $\left|\frac{I_{w_n}}{I_{\tilde{w}_n}}\right| \leq M$ for any adjacent intervals



$\boxed{\text{prop 4:}}$ $h : [0,1] \to [0,1]$ is M-QS.

h maps partitions pts of q to partitions pts of f.

Then f has bounded nearby geometry.

idea: on level n:

$$\frac{1}{M} \leq \frac{h(L)}{h(R)} \leq M$$

Def 3: A circle endomorphism f is called uniformly quasisymmetric if there exists a constant $M \geq 1$ such that

$$\frac{1}{M} \leq \frac{F^{-n}(x+t) - F^{-n}(x)}{F^{-n}(x) - F^{-n}(x-t)} \leq M, \quad \forall n \geq 1, \forall x \in \mathbb{R}, \forall t > 0.$$

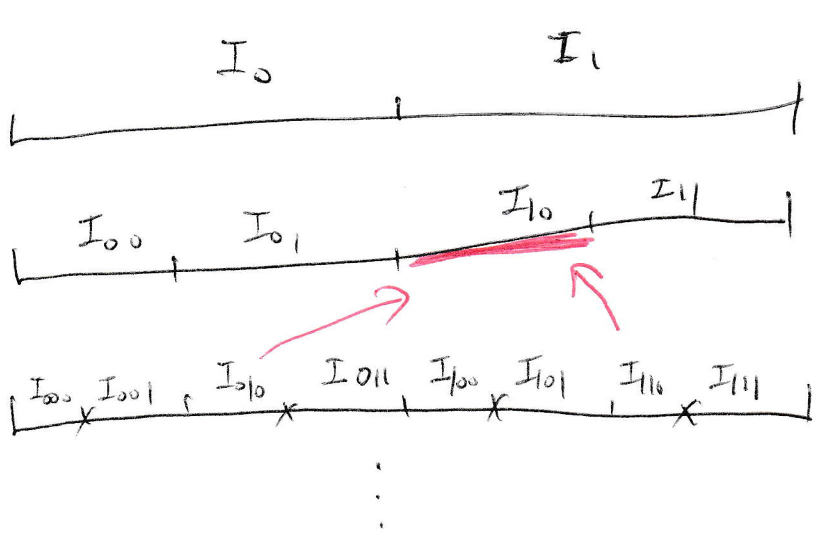
Prop 5. f is UQS $\Leftrightarrow$ bounded nearby geometry $\Leftrightarrow$ h is qs
where $f = h \circ q \circ h^{-1}$

Def 4: We say a circle endomorphism f preserves the Lebesgue measure m if

$m(f^{-1}(A)) = m(A)$ holds for all Borel subsets $A \subseteq S^1$.

For degree 2 的 Lebesgue invariant f:

$$|I_{w_n}| = |I_{0w_n}| + |I_{1w_n}|$$



$$|I_{10}| = |I_{010}| + |I_{110}|$$

$\boxed{\text{Prop } 6^*}$ If f is UQS and Lebesgue invariant, then the limit $\lim_{n\to\infty} \frac{|I_{w_n}|}{|I_{w_n 0}|}$ and $\lim_{n\to\infty} \frac{|I_{w_n}|}{|I_{w_n 1}|}$ exists along almost all dynamical path.

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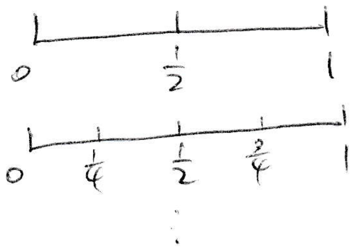
example 1: $q(z) = z^2$



$\frac{|I_{w_n}|}{|I_{w_n 0}|} = 2$, $\frac{|I_{w_n}|}{|I_{w_n 1}|} = 2$ for any w_n.

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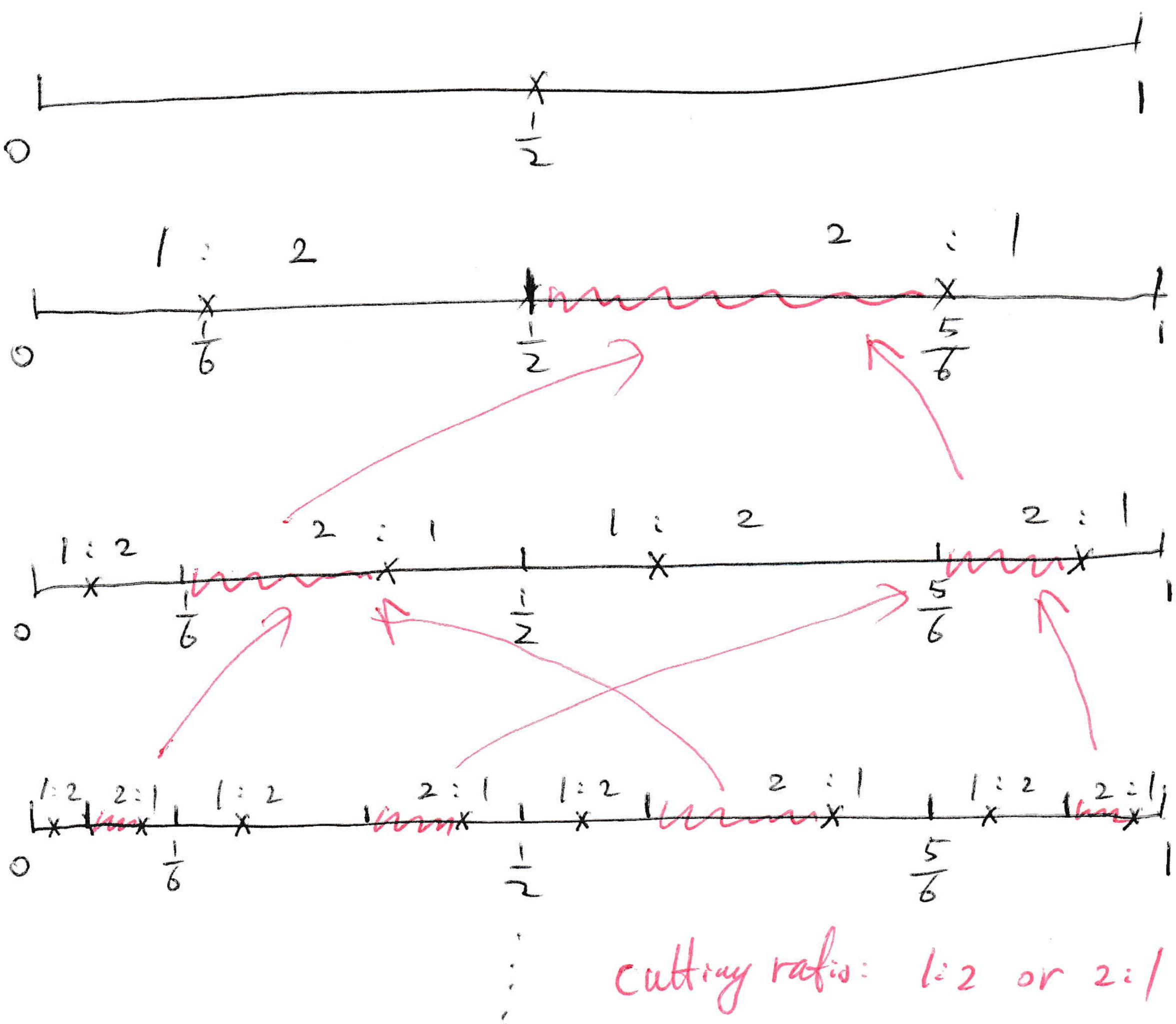
example 2: $f(x) = \begin{cases} 3x & x \in [0, \frac{1}{6}] \\ \frac{3}{2}x + \frac{1}{4} & x \in (\frac{1}{6}, \frac{1}{2}] \\ \frac{3}{2}x + \frac{1}{4} - 1 & x \in (\frac{1}{2}, \frac{5}{6}] \\ 3x - 2 & x \in (\frac{5}{6}, 1] \end{cases}$

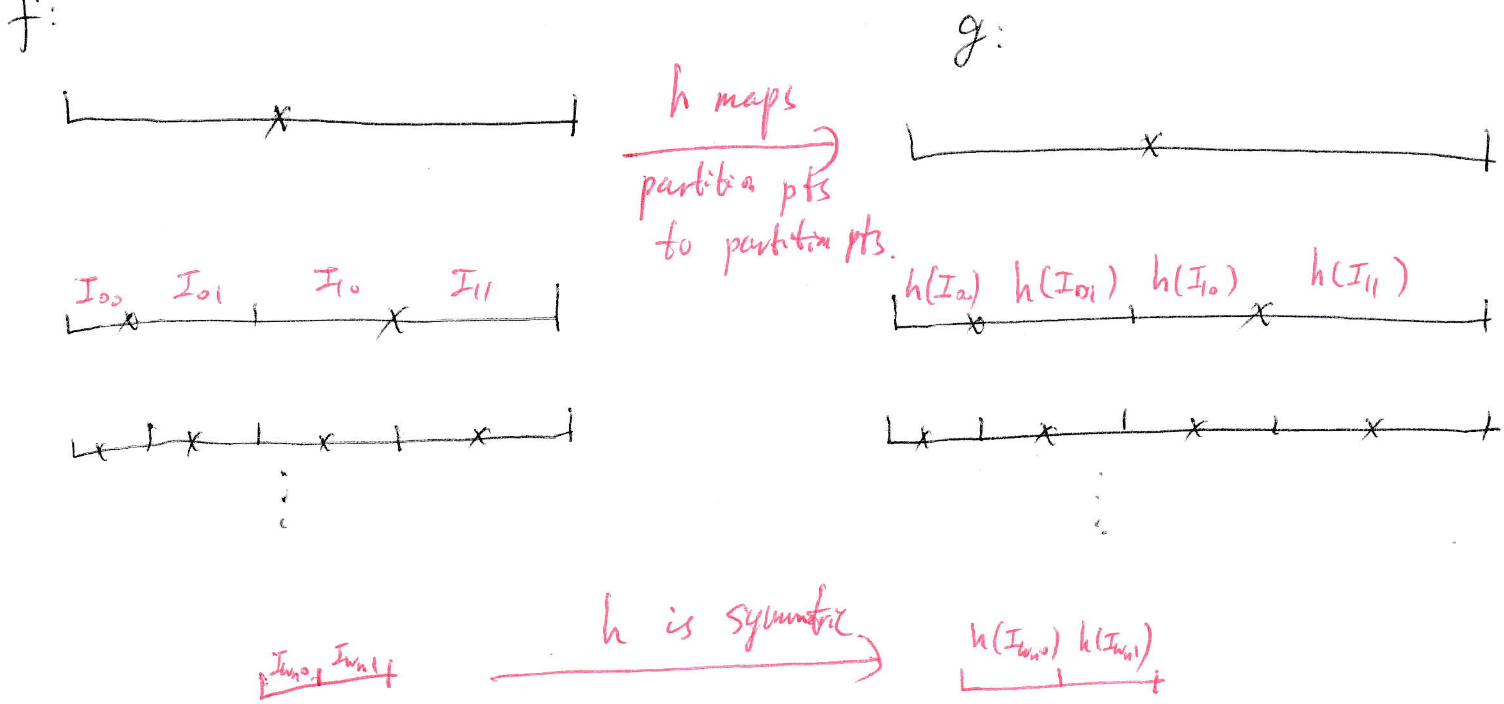


on red interval $I_{w_n 0}$

$\frac{|I_{w_n}|}{|I_{w_n 0}|} = \frac{3}{2}$ and $\left|\frac{I_{w_n}}{I_{w_n 1}}\right| = \frac{3}{1}$

cutting ratio: 1:2 or 2:1

$\boxed{\text{prop 7:}}$ suppose f and g are both UQS and Lebesgue invariant. If $f = h \circ g \circ h^{-1}$ and h is symmetric, then partitions of f and partitions of g have same ~~limits~~ "$\frac{\text{father}}{\text{son}}$" limits along almost all dynamical path.

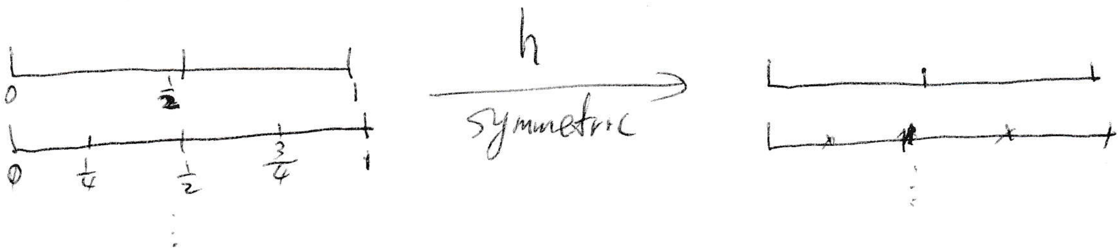


$$\frac{\text{father}}{\text{son}} \quad \frac{|I_{wn}|}{|I_{wn0}|}\left(\text{or } \frac{|I_{wn}|}{|I_{wn1}|}\right) \approx \frac{|h(I_{wn})|}{|h(I_{wn0})|}\left(\text{or } \frac{h(I_{wn})}{h(I_{wn1})}\right)$$

Main theorem of the new paper:
Let f and g be two circle endomorphisms of degree $d \geq 2$ such that each has bounded geometry, preserves the Lebegue measure and fixes 1. suppose $f = h^{-1} \circ g \circ h$, then h is symmetric $\iff h = Id$.

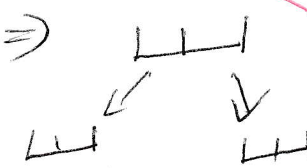
special cases:

① $g(z) = z^2$:



limit $\frac{\text{father}}{\text{son}} = 2$ limit $\frac{\text{father}}{\text{son}} = 2$

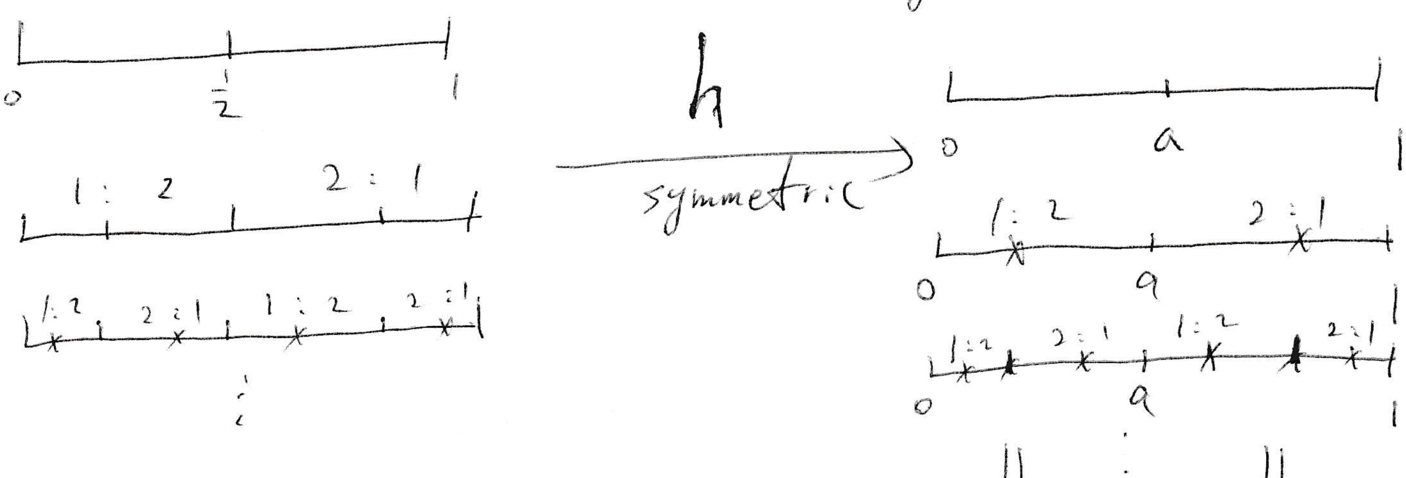
But Leb-invariant $\Rightarrow$



diverge.

② f:

g:



$\Downarrow$ same cutting + dense implies linear $\Downarrow$ same cutting + dense implies linear.

$m_1 x$ $m_2 x$

h is symmetric $\Rightarrow m_1 = m_2$.

part 3: symmetric at a point.

Sullivan's result: suppose f and g are two $C^{1+\text{Lipschitz}}$ circle expanding endomorphisms of the same degree. Let h be the conjugacy between f and g, that is, $f \circ h = h \circ g$. Then

h is $C^{1+\text{Lipschitz}}$ $\Leftrightarrow$ h is differentiable at one point with non-zero derivative.

Jiang's result: f and g are two $C^{1+\alpha}$ expanding endomorphisms of the same degree for $0<\alpha<1$. Let h be the conjugacy between f and g, that is, $f \circ h = h \circ g$. Then

h is $C^{1+\alpha}$ $\Leftrightarrow$ h is differentiable at one point with uniform bound.

Question: ~~sy h is sym For conjugat~~

For conjugacy map h, h is symmetric at a point $\overset{?}{\Leftrightarrow}$ h is symmetric.

Answer is No! (Hw)

$$f(z) = z\,\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}, \qquad g(z) = z^2$$

$f = h \circ g \circ h^{-1}$, h is symmetric at 1.

But h is not symmetric.