SYMMETRIC RIGIDITY FOR CIRCLE ENDOMORPHISMS HAVING BOUNDED GEOMETRY

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Abstract. Let f and g be two circle endomorphisms of degree $d \geq 2$ such that each has bounded geometry, preserves the Lebesgue measure, and fixes 1. Let h fixing 1 be the topological conjugacy from f to g. That is, $h \circ f = g \circ h$. We prove that h is a symmetric circle homeomorphism if and only if $h = Id$.

1. Introduction

A remarkable rigidity result in geometry is the Mostow rigidity theorem for high-dimensional closed hyperbolic manifolds (see [12]). For low-dimensional hyperbolic manifolds, the situation is quite complicated and the study of the rigidity problem can be used by one-dimensional dynamical systems. In this paper, we study the symmetric rigidity problem in one-dimensional dynamical systems, which is motivated by the study of geometric Gibbs theory (see [9–11]). The main result is that

Theorem 1 (Main Theorem)**.** *Suppose f and g are two circle endomorphisms of the same degree $d \geq 2$ having bounded geometry such that $f(1) = g(1) = 1$ and suppose f and g both preserve the Lebesgue measure on the unit circle. Let h be the conjugacy from f to g with $h(1) = 1$. That is, $h \circ f = g \circ h$. If h is a symmetric homeomorphism, then h must be the identity.*

The result has many consequences, in particular, it gives an affirmative answer to [11, Conjecture 2.4] and [10, Conjecture10.12] and [5, Conjecture 2] as follows.

Corollary 1. *Suppose f and g are uniformly symmetric circle endomorphisms of the same degree d and and suppose both f and g preserve the Lebesgue measure m. Suppose h is the conjugacy from f to g, and $h(1) = 1$. If h is symmetric, then $h = id$.*

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Corollary 2. *Suppose f and g are uniformly quasisymmetric circle endomorphisms of the same degree d and both f and g preserve the Lebesgue measure m. Suppose h is the conjugacy from f to g, and $h(1) = 1$. If h is symmetric, then $h = id$.*

We organize this paper as follows. In Section 2, we define a circle endomorphism having bounded geometry. In the same section, we review the definitions of uniformly quasisymmetric circle endomorphisms and uniformly symmetric circle endomorphisms. In Section 3, we prove our main theorem (Theorem 1).

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2. Circle Endomorphisms Having Bounded Geometry

Let $T = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle in the complex plane $\mathbb{C}$. Let m be the Lebesgue probability measure on T (i.e. a Haar measure on T). Suppose

$$f : T \to T$$

is an orientation-preserving covering map of degree $d \geq 2$. We call it a circle endomorphism. Suppose

$$h : T \to T$$

is an orientation-preserving homeomorphism. We call it a circle homeomorphism. Every circle endomorphism f has at least one fixed point. By conjugating f by a rotation of the circle if necessary, we assume that 1 is a fixed point of f, that is, $f(1) = 1$.

The universal cover of T is the real line $\mathbb{R}$ with a covering map

$$\pi(x) = e^{2\pi ix} : \mathbb{R} \to T.$$

In this way, we can think the unit interval $[0, 1]$ as the unit circle T.

Then every circle endomorphism f can be lifted to a homeomorphism

$$F : \mathbb{R} \to \mathbb{R}, \quad F(x + 1) = F(x) + d, \quad \forall x \in \mathbb{R}.$$

We will assume that $F(0) = 0$ so that there is a one-to-one correspondence between f and F. Therefore, we also call such a map F a circle endomorphism.

Similarly, every circle homeomorphism h can be lifted to an orientation-preserving homeomorphism

$$H : \mathbb{R} \to \mathbb{R}, \quad H(x + 1) = H(x) + 1, \quad \forall x \in \mathbb{R}.$$

We will assume that $0 \leq H(0) < 1$ so that there is a one-to-one correspondence between h and H. Therefore, we also call such a map H a circle homeomorphism. Since we only consider circle homeomorphisms as conjugacies of circle endomorphisms in this paper, we assume $h(1) = 1$ (equivalently, $H(0) = 0$). We use id and ID to denote the identity circle homeomorphism and its lift to $\mathbb{R}$, respectively. That is,

$id(z) = z$ and $ID(x) = x$. Henceforth, we assume that $d \geq 2$ is fixed and will not mention it always.

Definition 1. *A circle homeomorphism h is called quasisymmetric (refer to [2]) if there exists a constant $M \geq 1$ such that*

$$\frac{1}{M} \leq \frac{H(x+t) - H(x)}{H(x) - H(x-t)} \leq M \quad \forall x \in \mathbb{R}, \ \forall t > 0.$$

It is called symmetric (refer to [3]) if there exists a positive bounded function $\epsilon(t)$ such that $\epsilon(t) \to 0$ as $t \to 0^+$ and

$$\frac{1}{1+\epsilon(t)} \leq \frac{H(x+t) - H(x)}{H(x) - H(x-t)} \leq 1 + \epsilon(t) \quad \forall x \in \mathbb{R}, \ \forall t > 0.$$

Definition 2. *A circle endomorphism f is called uniformly quasisymmetric (refer to [8, 10]) if there exists a constant $M \geq 1$ such that*

$$\frac{1}{M} \leq \frac{F^{-n}(x+t) - F^{-n}(x)}{F^{-n}(x) - F^{-n}(x-t)} \leq M \quad \forall n \geq 1, \ \forall x \in \mathbb{R}, \ \forall t > 0.$$

It is called uniformly symmetric (refer to [10]) if there exists a positive bounded function $\epsilon(t)$ such that $\epsilon(t) \to 0$ as $t \to 0^+$ and

$$\frac{1}{1+\epsilon(t)} \leq \frac{F^{-n}(x+t) - F^{-n}(x)}{F^{-n}(x) - F^{-n}(x-t)} \leq 1 + \epsilon(t) \quad \forall n \geq 1, \ \forall x \in \mathbb{R}, \ \forall t > 0.$$

An example of a uniformly symmetric circle endomorphism is a C^{1+Dini} (or $C^{1+\alpha}$) expanding circle endomorphism (see [10]).

Definition 3. *We say a circle endomorphism f preserves the Lebesgue measure m if*

$$m(f^{-1}(A)) = m(A) \tag{1}$$

holds for all Borel subsets $A \subseteq T$.

Henceforth, in order to avoid confusion, we will consistently use

$$[0,1]/\{0 \sim 1\} = \mathbb{R} \pmod 1$$

to mean the unit circle. Likewise, we will consistently use

$$f = F \pmod 1 : [0,1]/\{0 \sim 1\} \to [0,1]/\{0 \sim 1\}$$

to mean a circle endomorphism and

$$h = H \pmod 1 : [0,1]/\{0 \sim 1\} \to [0,1]/\{0 \sim 1\}$$

to mean a circle homeomorphism.

For any circle endomorphism f, the preimage $f^{-1}(0)$ of the fixed point 0 partitions $[0,1]$ into d closed and ordered intervals I_0, I_1, $\cdots$, I_{d-1} (see Figure 1). Let

$$\eta_1 = \{I_0, I_1, \cdots, I_{d-1}\}.$$

Then η_1 is a Markov partition. That is,

(i) $[0,1] = \cup_{i=0}^{d-1} I_i$;
(ii) I_i and I_j have pairwise disjoint interiors for any $0 \leq i < j \leq d-1$;
(iii) $f(I_i) = [0,1]$ for every $0 \leq i \leq d-1$;
(iv) the restriction of f to the interior of I_i is injective for every $0 \leq i \leq d-1$.

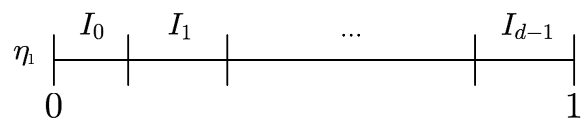

FIGURE 1. The initial Markov partition.

The preimage $f^{-n}(0)$ of the fixed point 0 partitions $[0,1]$ into d^n closed intervals I_{w_n} labeled by

$$w_n = i_0 i_1 \ldots i_{n-1} \in \Sigma_n = \prod_{k=0}^{n-1} \{0, 1, \ldots, d-1\}$$

and defined inductively as

$$f^k(I_{w_n}) \subset I_{i_k}, \ \ \forall 0 \leq k \leq n-2, \quad \text{and} \quad f^{n-1}(I_{w_n}) = I_{i_{n-1}}.$$

Let

$$\eta_n = \{ I_{w_n} \mid w_n = i_0 i_1 \ldots i_{n-1} \in \Sigma_n \}.$$

Then η_n is also a Markov partition. That is,

(1) $[0,1] = \cup_{w_n \in \Sigma_n} I_{w_n}$;
(2) intervals in η_n have pairwise disjoint interiors;
(3) $f^n(I_{w_n}) = [0,1]$ for every $w_n \in \Sigma_n$;
(4) the restriction of f^n to the interior of I_{w_n} is injective for every $w_n \in \Sigma_n$.

Remark 1. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are two partitions of T. The partition*

$$A \vee B = \{ A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B} \}$$

is the finer partition from $\mathcal{A}$ and $\mathcal{B}$. Then we have that

$$\eta_n = \vee_{k=1}^{n} f^{-k} \eta_1.$$

Let σ be the left-shift map and let σ^* be the right-shift map on Σ_n, that is,

$$\sigma(\omega_n) = \sigma(i_0 i_1 \ldots i_{n-2} i_{n-1}) = i_1 \ldots i_{n-2} i_{n-1}$$

and

$$\sigma^*(\omega_n) = \sigma^*(i_0 i_1 \ldots i_{n-2} i_{n-1}) = i_0 i_1 \ldots i_{n-2}.$$

Here we assume $w_0 = \emptyset$ and $I_{w_0} = [0,1]$ and $\sigma(w_1) = w_0$ and $\sigma^*(w_1) = w_0$. Then we have

$$I_{w_n} = \cup_{k=0}^{d-1} I_{w_n k} = \cup_{w_{n+1} \in (\sigma^*)^{-1}(w_n)} I_{w_{n+1}}$$

and

$$f^{-1}(I_{w_n}) = \cup_{k=0}^{d-1} I_{k \omega_n} = \cup_{w_{n+1} \in \sigma^{-1}(w_n)} I_{w_{n+1}}.$$

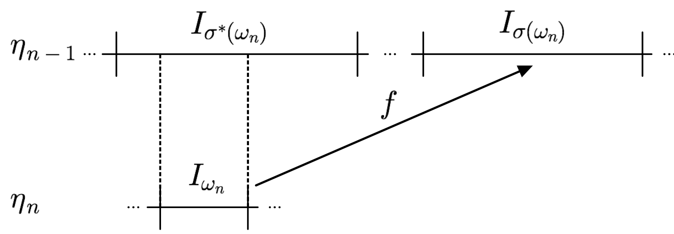

FIGURE 2. $I_{w_n} \subset I_{\sigma^*(w_n)}$ and $f(I_{w_n}) = I_{\sigma(w_n)}$.

Definition 4. *A circle endomorphism f is said to have bounded geometry (refer to [8, 10]) if there is a constant $C > 1$ such that*

(2) $$\frac{|I_{\sigma^*(\omega_n)}|}{|I_{\omega_n}|} \leq C, \quad \forall \omega_n \in \Sigma_n, \;\; \forall n \geq 1. \quad \textit{(See Figure 2)}$$

We know that a uniformly quasisymmetric circle endomorphism as well as a uniformly sisymmetric circle endomorphism has bounded geometry (refer to [8,10]). However, the converse is not true. For example, for any $\alpha \in (1/2, 1)$, the piecewise-linear degree 2 circle endomorphism

$$f_\alpha(x) = \begin{cases} f_\alpha(x+1) - 2 & \text{if } x < 0 \\ x/\alpha & \text{if } 0 \leq x < \alpha \\ 1 + (x-\alpha)/(1-\alpha) & \text{if } \alpha \leq x < 1 \\ f_\alpha(x-1) + 2 & \text{if } 1 \leq x \end{cases}$$

has bounded geometry because

$$\frac{|I_{\sigma^*(\omega_n)}|}{|I_{\omega_n}|} \in \{1/\alpha, 1/(1-\alpha)\}, \quad \forall \omega_n \in \Sigma_n, \forall n \geq 1.$$

However, f_α is not uniformly quasisymmetric since for any $0 < t < 1 - \alpha$,

$$\frac{f_\alpha^{-n}(0+t) - f_\alpha^{-n}(0)}{f_\alpha^{-n}(0) - f_\alpha^{-n}(0-t)} = \left(\frac{\alpha}{1-\alpha}\right)^n \to \infty \text{ as } n \to \infty.$$

Remark 2. *The property of uniform quasisymmetry for a circle endomorphism can be equivalently characterized in terms of its sequence of nested partitions $\{\eta_n\}$ alone by saying that the circle enedomorphism has bounded nearby geometry. The precise definition of bounded nearby geometry is given and its equivalence to uniform quasisymmetry is proved in [6, 7]. For more on circle endomorphisms with bounded geometry and/or bounded nearby geometry, see also [1, 4, 8, 10]).*

For a circle endomorphism f having bounded geometry, let

$$\tau_n = \max\{|I_{w_n}| \mid w_n \in \Sigma_n\}.$$

Then from Definition 4, we have a constant $0 < \tau < 1$ such that

(3) $$\tau_n \leq \tau^n, \quad \forall n \geq 1.$$

It follows that any two circle endomorphisms f and g having bounded geometry are topologically conjugate. That is, there is a circle homoemorphism h such that

$$f \circ h = h \circ g. \tag{4}$$

Here h is called the conjugacy from f to g, and when h is symmetric we call it a symmetric conjugacy. In the special case that both f and g are uniformly quasisymmetric, we further know that the conjugacy h is quasisymmetric. However, as long as at least one of the maps is not uniformly quasisymmetric, the conjugacy h may not be quasisymmetric. Refer to [1, 4, 5, 8, 10].

3. Symmetric Rigidity, the Proof of the Main Result

We start with the following lemma.

Lemma 1. *Suppose f is a circle endomorphism having bounded geometry. Let $\{\eta_n\}_{n=1}^{\infty}$ be the corresponding sequence of partitions. Suppose $I_{w_n} \in \eta_n$ is a fixed partition interval for some $n \geq 1$. Then*

$$\lim_{k\to\infty} \sum_{w_n^1 \neq w_n} \cdots \sum_{w_n^k \neq w_n} |I_{w_n^1 \cdots w_n^k}| = 0,$$

where the ω_n^i are all words of length n.

Proof. From the definition of bounded geometry (Definition 4), we have that

$$|I_{w_n}| \geq A = \frac{1}{C^n}.$$

Since

$$\bigcup_{w_n^1 \neq w_n} I_{w_n^1} = \overline{[0,1] \setminus I_{w_n}},$$

we get

$$\sum_{w_n^1 \neq w_n} |I_{w_n^1}| = 1 - |I_{w_n}| \leq 1 - A.$$

For any w_n^1, we have that $I_{w_n^1 w_n} \subset I_{w_n^1}$. Because of bounded geometry, we further have

$$|I_{w_n^1 w_n}| \geq A|I_{w_n^1}|.$$

Since

$$\bigcup_{w_n^2 \neq w_n} I_{w_n^1 w_n^2} = \overline{I_{w_n^1} \setminus I_{w_n^1 w_n}},$$

we have

$$\sum_{w_n^2 \neq w_n} |I_{w_n^1 w_n^2}| = |I_{w_n^1}| - |I_{w_n^1 w_n}| \leq |I_{\omega_n^1}| - A|I_{\omega_n^1}| = (1-A)|I_{w_n^1}|.$$

This implies that

$$\sum_{w_n^1 \neq w_n} \sum_{w_n^2 \neq w_n} |I_{w_n^1 w_n^2}| \leq (1-A) \sum_{w_n^1 \neq w_n} |I_{w_n^1}| \leq (1-A)^2.$$

Inductively, suppose we know

$$\sum_{w_n^1 \neq w_n} \cdots \sum_{w_n^{k-1} \neq w_n} |I_{w_n^1 \ldots w_n^{k-1}}| \leq (1-A)^{k-1}$$

for $k \geq 3$. Then

$$\sum_{w_n^1 \neq w_n} \cdots \sum_{w_n^k \neq w_n} |I_{w_n^1 \ldots w_n^k}| = \sum_{w_n^1 \neq w_n} \cdots \sum_{w_n^{k-1} \neq w_n} \Big(|I_{w_n^1 \cdots w_n^{k-1}}| - |I_{w_n^1 \cdots w_n^{k-1} w_n}| \Big)$$

$$= \sum_{w_n^1 \neq w_n} \cdots \sum_{w_n^{k-1} \neq w_n} |I_{w_n^1 \cdots w_n^{k-1}}| \Big(1 - \frac{|I_{w_n^1 \cdots w_n^{k-1} w_n}|}{|I_{w_n^1 \cdots w_n^{k-1}}|} \Big).$$

Notice that $I_{w_n^1 \cdots w_n^{k-1} w_n} \subset I_{w_n^1 \cdots w_n^{k-1}}$. The definition of bounded geometry implies that

$$\frac{|I_{w_n^1 \cdots w_n^{k-1} w_n}|}{|I_{w_n^1 \cdots w_n^{k-1}}|} \geq A.$$

This implies that

$$1 - \frac{|I_{w_n^1 \cdots w_n^{k-1} w_n}|}{|I_{w_n^1 \cdots w_n^{k-1}}|} \leq 1 - A.$$

Thus

$$\sum_{w_n^1 \neq w_n} \cdots \sum_{w_n^k \neq w_n} |I_{w_n^1 \ldots w_n^k}| \leq (1-A) \sum_{w_n^1 \neq w_n} \cdots \sum_{w_n^{k-1} \neq w_n} |I_{w_n^1 \cdots w_n^{k-1}}| \leq (1-A)^k.$$

Letting $k \to \infty$, this proves the lemma. □

Given a partition interval $I_{w_n} \in \eta_n$ for some $n \geq 1$, define

$$C(I_{w_n}) = \{x \in [0,1] \mid f^{kn}(x) \notin I_{w_n}, k = 0, 1, 2, \cdots\} = \bigcap_{i=1}^{\infty} \left(\bigcup_{\substack{\omega_n^j \neq \omega_n \\ 1 \leq j \leq i}} I_{\omega_n^1 \cdots \omega_n^i} \right).$$

A consequence of Lemma 1 is the following.

Corollary 3. *Suppose f is a circle endomorphism having bounded geometry. Then the set $C(I_{w_n})$ has zero Lebesgue measure. That is, $m(C(I_{w_n})) = 0$.*

Suppose f and g are both circle endomorphisms having bounded geometry and h is the conjugacy from f to g. Define the number

$$1 \leq \Phi = \sup_{I \subseteq [0,1]} \frac{|h(I)|}{|I|} \leq \infty$$

and the set

$$X = \{x \in [0,1] \mid \exists I_k^x = [a_k, b_k], \lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = x, \lim_{k\to\infty} \frac{|h(I_k^x)|}{|I_k^x|} = \Phi\} \tag{5}$$

We would like to note that, in general, $\Phi = \infty$ and when $\Phi < \infty$, h is a Lipschitz conjugacy.

Remark 3. *Similarly, we can also define*

$$0 \leq \phi = \inf_{I \subseteq [0,1]} \frac{|h(I)|}{|I|} \leq 1$$

and use ϕ *to prove Theorem 1.*

Lemma 2. *Suppose* f *and* g *are both circle endomorphisms having bounded geometry. Then* X *is a non-empty subset of* T*.*

Proof. Suppose $\{I_k = [a_k, b_k]\}_{k=1}^{\infty}$ is a sequence of intervals such that

$$\lim_{k\to\infty} \frac{|h(I_k)|}{|I_k|} = \Phi.$$

By taking a subsequence if necessary, we assume that $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ are two convergent sequences of numbers and $a = \lim_{k\to\infty} a_k$ and $b = \lim_{k\to\infty} b_k$.

If $a = b = x$, then $x \in X$ and $X \neq \emptyset$. Note that if $\Phi = \infty$ then $a = b$.

If $a < b$, then $I = [a, b]$ is a non-trivial interval such that

$$\frac{|h(I)|}{|I|} = \Phi. \tag{6}$$

In this case, we claim that for any non-trivial subinterval $I' \subset I$, $|h(I')|/|I'| = \Phi$. The claim implies that $I \subset X$, and thus, $X \neq \emptyset$. Now we prove the claim as follows. Let $I' = [a', b']$ with $a \leq a' < b' \leq b$. Let $L = [a, a']$ and $R = [b', b]$. Then we have $I = L \cup I' \cup R$ and $h(I) = h(L) \cup h(I') \cup h(R)$. Assume $|h(I')|/|I'| < \Phi$. Then, since $|h(L)| \leq \Phi|L|$, and $|h(R)| \leq \Phi|R|$, we have

$$\frac{|h(I)|}{|I|} = \frac{|h(L)| + |h(I')| + |h(R)|}{|L| + |I'| + |R|} < \Phi.$$

This is a contradiction. Thus we have proved the claim and completed the proof. □

Furthermore, under the assumption that both f and g preserve the Lebesgue measure m, we have the following stronger result.

Lemma 3. *Suppose* f *and* g *are both circle endomorphisms having bounded geometry and both preserve the Lebesgue measure* m*. Then* X *is dense in* $[0, 1]$*. That is,* $\overline{X} = [0, 1]$*.*

Proof. We will prove that for any $n \geq 1$ and for any partition interval $I_{w_n} \in \eta_n$, $I_{w_n} \cap X \neq \emptyset$. It will then follow from inequality (3) that $\overline{X} = [0, 1]$. We prove it by contradiction.

Assume we have a partition interval I_{w_n} such that $I_{w_n} \cap X = \emptyset$. Then we can find a number $D < \Phi$ such that

$$\frac{|h(I)|}{|I|} \leq D \tag{7}$$

for all $I \subset I_{w_n}$.

Since $X \neq \emptyset$, we have an interval $I^D \subseteq [0,1]$ such that

$$\frac{|h(I^D)|}{|I^D|} > D. \tag{8}$$

We pull back I^D by f^n to get $f^{-n}(I^D) = \cup_{w_n^1} I^D_{w_n^1}$, where $I^D_{w_n^1} \subset I_{w_n^1} \in \eta_n$ and $f^n(I^D_{w_n^1}) = I^D$.

Since both f and g preserve the Lebesgue measure m, for all $k \geq 2$ we have

$$\begin{aligned}
|I^D| &= |I^D_{\omega_n}| + \sum_{\omega_n^1 \neq \omega_n} |I^D_{\omega_n^1}| = |I^D_{\omega_n}| + \sum_{\omega_n^1 \neq \omega_n} |f^{-n}(I^D_{\omega_n^1})| \\
&= |I^D_{\omega_n}| + \sum_{\omega_n^1 \neq \omega_n} \left(|I^D_{\omega_n \omega_n^1}| + \sum_{\omega_n^2 \neq \omega_n} |I^D_{\omega_n^2 \omega_n^1}| \right) \\
&= |I^D_{\omega_n}| + \sum_{\omega_n^1 \neq \omega_n} |I^D_{\omega_n \omega_n^1}| + \sum_{\omega_n^2 \neq \omega_n} \sum_{\omega_n^1 \neq \omega_n} |I^D_{\omega_n^2 \omega_n^1}| = \dots \\
&= |I^D_{w_n}| + \sum_{l=1}^{k-1} \sum_{w_n^l \neq w_n} \cdots \sum_{w_n^1 \neq w_n} |I^D_{w_n w_n^l \dots w_n^1}| + \sum_{w_n^k \neq w_n} \cdots \sum_{w_n^1 \neq w_n} |I^D_{w_n^k \dots w_n^1}| \qquad (9)
\end{aligned}$$

and, similarly,

(10)

$$|h(I^D)| = |h(I^D_{w_n})| + \sum_{l=1}^{k-1} \sum_{w_n^l \neq w_n} \cdots \sum_{w_n^1 \neq w_n} |h(I^D_{w_n w_n^l \dots w_n^1})| + \sum_{w_n^k \neq w_n} \cdots \sum_{w_n^1 \neq w_n} |h(I^D_{w_n^k \dots w_n^1})|.$$

See Figure 3. Because $I^D_{w_n}$ and $I^D_{w_n w_n^l \dots w_n^1}$ are sub-intervals of I_{w_n}, (7) says that

$$\frac{|h(I^D_{w_n})|}{|I^D_{w_n}|}, \quad \frac{|h(I^D_{w_n w_n^l \cdots w_n^1})|}{|I^D_{w_n w_n^l \cdots w_n^1}|} \leq D \ \ \forall \, l \geq 1.$$

This implies that

$$\frac{|h(I^D_{w_n})| + \sum_{l=1}^{k-1} \sum_{w_n^l \neq w_n} \cdots \sum_{w_n^1 \neq w_n} |h(I^D_{w_n w_n^l \dots w_n^1})|}{|I^D_{w_n}| + \sum_{l=1}^{k-1} \sum_{w_n^l \neq w_n} \cdots \sum_{w_n^1 \neq w_n} |I^D_{w_n w_n^l \dots w_n^1}|} \leq D \ \ \forall \, k \geq 2. \tag{11}$$

From Lemma 1

$$\lim_{k \to \infty} \sum_{w_n^k \neq w_n} \cdots \sum_{w_n^1 \neq w_n} |I^D_{w_n^k \cdots w_n^1}| = 0 \tag{12}$$

and

$$\lim_{k \to \infty} \sum_{w_n^k \neq w_n} \cdots \sum_{w_n^1 \neq w_n} |h(I^D_{w_n^k \cdots w_n^1})| = 0. \tag{13}$$

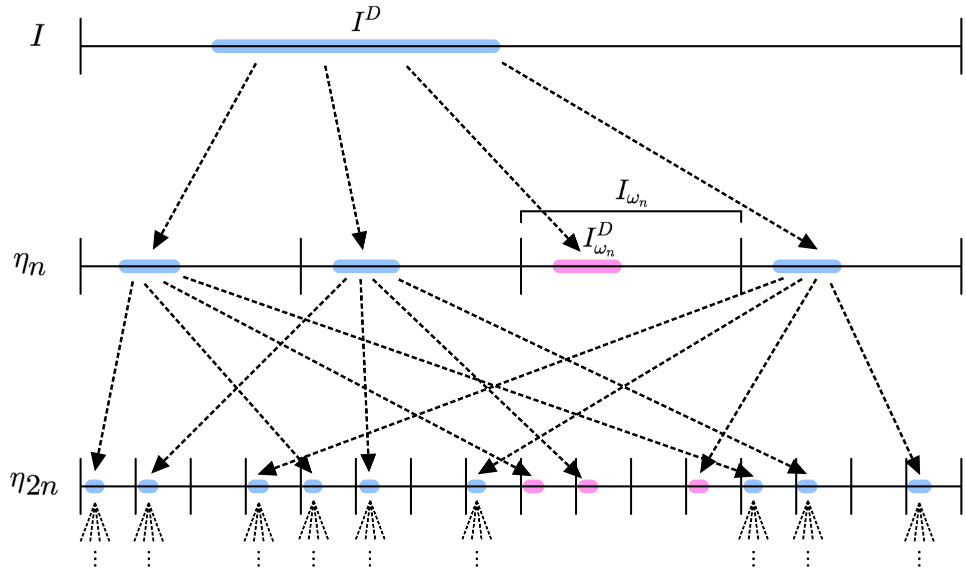

FIGURE 3. The interval I^D has a preimage under f^n composed of d^n intervals, one of which is a subset of I_{ω_n}. Similarly, each of these preimage-intervals that is not a subset of I_{ω_n} has a preimage under f^n composed of d^n intervals, one of which is a subset of I_{ω_n}. Equation (9) says that the length of I^D is equal to the sum of the lengths of all blue intervals belonging to the same arbitrary level plus the lengths all pink intervals belonging to that same level or any previous level.

Now (9), (10), (11), (12), and (13) imply that

$$\frac{|h(I^D)|}{|I^D|} \leq D.$$

This contradicts (8). Thus our assumption that there exists a partition interval I_{ω_n} such that $I_{\omega_n} \cap X = \emptyset$ is false, and this proves the lemma. □

Proof of Theorem 1. We will prove that $\Phi = 1$. Equivalently, we will prove that $\Phi > 1$ cannot happen, regardless of $\Phi < \infty$ or $\Phi = \infty$.

We proceed with a proof by contradiction. Assume $\Phi > 1$ (possibly ∞). Then we have two numbers $1 < D_1 < D_2 < \Phi$. Since h is symmetric (see definition 1), there exists a positive bounded function $\epsilon(t)$ such that $\epsilon(t) \to 0$ as $t \to 0^+$ and

$$\frac{1}{1+\epsilon(t)} \leq \frac{|h(I)|}{|h(I')|} \leq 1 + \epsilon(t)$$

holds for all closed intervals I and I' that have the same length $t > 0$ and are adjacent, i.e. the right endpoint of one interval is the left endpoint of the other. Fix t_0 such that

$$\epsilon(t) < \frac{D_2}{D_1} - 1 \quad \forall t < t_0. \tag{14}$$

Since $\overline{X} = [0,1]$ (Lemma 3), there exists an interval $I = [a,b] \subset (0,1)$ with $|I| = b - a < t_0$ such that

$$\frac{|h(I)|}{|I|} > D_2. \tag{15}$$

Let $L = [2a-b, a] \subset (0,1)$ and $R = [b, 2b-a] \subset (0,1)$ (see figure 4). Then the intervals L and R are adjacent to I and have the same length as $|I|$. It follows from (14) that

$$\frac{|h(R)|}{|R|} = \frac{|h(R)|}{|h(I)|} \cdot \frac{|h(I)|}{|I|} \cdot \frac{|I|}{|R|} > \frac{1}{1+\epsilon(b-a)} \cdot D_2 \cdot 1 > D_1$$

and

$$\frac{|h(L)|}{|L|} = \frac{|h(L)|}{|h(I)|} \cdot \frac{|h(I)|}{|I|} \cdot \frac{|I|}{|L|} > \frac{1}{1+\epsilon(b-a)} \cdot D_2 \cdot 1 > D_1.$$

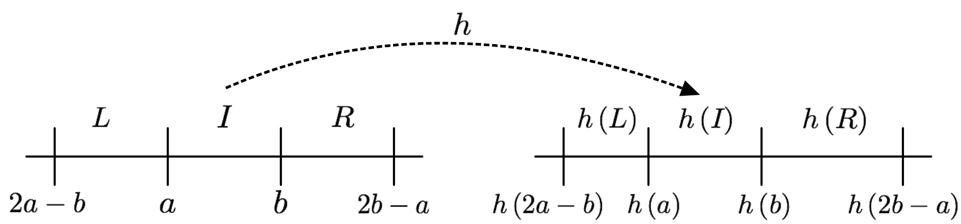

FIGURE 4. h is symmetric.

Now we want to show that

$$\frac{|h([a,1])|}{|[a,1]|} > D_1.$$

Consider any interval $J = [b,c] \supset R$ with $2b - a \leq c \leq 1$ satisfying

$$\frac{|h(J)|}{|J|} > D_1. \tag{16}$$

If $c = 1$, we have

$$\frac{|h([a,1])|}{|[a,1]|} = \frac{|h(I \cup J)|}{|I \cup J|} > D_1.$$

Then we have nothing further to prove.

If $c < 1$, we have a number $\delta > 0$ such that $c + \delta < 1$ and such that for any $x \in [c, c+\delta]$ we have

$$\frac{|h([b,x])|}{|[b,x]|} > D_1.$$

Since $\overline{X} = [0,1]$ (Lemma 3), there is an interval $I_1 = [a_1, b_1] \subset [c, c+\delta]$ with $|I_1| < t_0$ such that

$$\frac{|h(I_1)|}{|I_1|} > D_2.$$

Let $J_1 = [b, a_1]$. Then we have three consecutive intervals I, J_1, and I_1 such that

$$\frac{|h([a,b_1])|}{|[a,b_1]|} = \frac{|h(I \cup J_1 \cup I_1)|}{|I \cup J_1 \cup I_1|} > D_1.$$

(See Figure 5.)

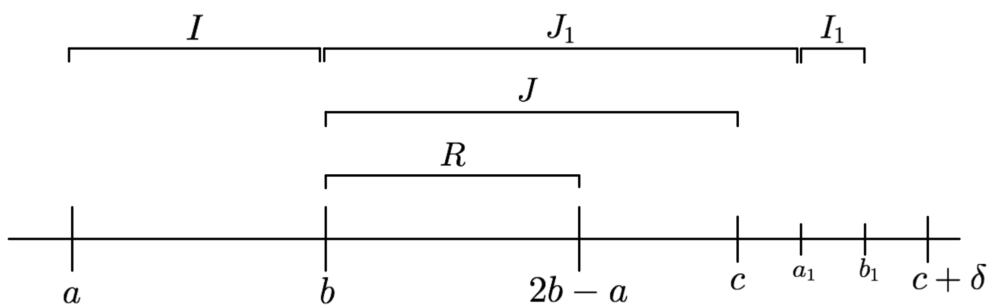

FIGURE 5. Construction of J_1 and I_1.

Consider I_1 as a new I and repeat the above construction. We get three consecutive intervals $I_1 = [a_1, b_1]$, $J_2 = [b_1, a_2]$, and $I_2 = [a_2, b_2]$ such that

$$\frac{|h([a_1,b_2])|}{|[a_1,b_2]|} = \frac{|h(I_1 \cup J_2 \cup I_2)|}{|I_1 \cup J_2 \cup I_2|} > D_1.$$

Inductively, for every integer $n \geq 2$, we have three consecutive intervals $I_{n-1} = [a_{n-1}, b_{n-1}]$, $J_n = [b_{n-1}, a_n]$, and $I_n = [a_n, b_n]$ such that

$$\frac{|h([a_{n-1},b_n])|}{|[a_{n-1},b_n]|} = \frac{|h(I_{n-1} \cup J_n \cup I_n)|}{|I_{n-1} \cup J_n \cup I_n|} > D_1.$$

This implies that

$$\frac{|h([a,b_n])|}{|[a,b_n]|} = \frac{|h(I \cup (\cup_{i=1}^{n}(J_i \cup I_i)))|}{|I \cup (\cup_{i=1}^{n}(J_i \cup I_i))|} > D_1.$$

If $b_n = 1$, we have

$$\frac{|h([a,1])|}{|[a,1]|} > D_1.$$

Then we have nothing further to prove.

In the case that $b_n < 1$ for all $n \geq 1$, since $\{b_n\}_{n=1}^{\infty}$ is a strictly increasing sequence in $[0,1)$, we have

$$b_\infty = \lim_{n\to\infty} b_n \leq 1.$$

and

$$\frac{|h([a,b_\infty])|}{|[a,b_\infty]|} = \frac{|h(I \cup (\cup_{n=1}^{\infty}(J_n \cup I_n)))|}{|I \cup (\cup_{n=1}^{\infty}(J_n \cup I_n))|} > D_1.$$

Since b_∞ depends on the initially chosen interval J, we write it as $b_\infty(J)$. Consider the set

$$\mathcal{B} = \{b_\infty(J) \mid J \text{ satisfies (16)}\}$$

Let $\beta = \sup \mathcal{B}$. We claim $\beta = 1$. Otherwise, we take $J = [b,\beta]$. It satisfies (16). Then $b_\infty(J) > \beta$. This contradiction proves the claim, and so

$$\frac{|h([a,1])|}{|[a,1]|} > D_1.$$

Similarly, by using L instead of R and applying the procedure above, we get

$$\frac{|h([0,a])|}{|[0,a]|} > D_1.$$

Finally, we get the following contradiction.

$$1 = \frac{|h([0,1])|}{|[0,1]|} = \frac{|h([0,a)| + |h([a,1])|}{|[0,a]| + |[a,1]|} > D_1 > 1.$$

The contradiction implies that $\Phi = 1$.

Since $\Phi = 1$, we have that for any non-trivial interval $J \subset [0,1]$, $|h(J)|/|J| = 1$. Otherwise, if there is an interval $J \subset [0,1]$ such that $|h(J)|/|J| < 1$, let $L \cup R = [0,1] \setminus J$. Then

$$1 = \frac{|h([0,1])|}{|[0,1]|} = \frac{|h(L)| + |h(J)| + |h(R)|}{|L| + |J| + |R|} < 1,$$

since $|h(L)| \leq |L|$ and $|h(R)| \leq |R|$. This is a contradiction. Since $h(0) = 0$, it follows that $h = id$. This completes the proof of Theorem 1. □

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