An Overview of Holomorphic Motion Theory

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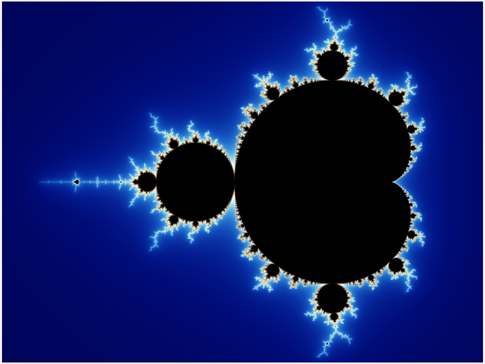
Starting

The study of holomorphic motions started from two independent papers (Mãné-Sad-Sullivan, 1983 and Lyubich, 1983). The original purpose is to use holomorphic motion in the study of the hyperbolicity conjecture in the dynamics of rational maps. However, up to today, the hyperbolicity conjecture is still open but by using the λ-Lemma in the study of holomorphic motions, they were able to prove that all structurally stable rational maps are generic (open and dense).

An example



The hyperbolicity conjecture: The set of hyperbolic quadratic polynomials $q_c(z) = z^2 + c$ is open and dense in the complex plane.

Definition of a holomorphic motion over the open unit disk

Let $\Delta = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$ be the open unit disk. Suppose E is a subset of the Riemann sphere $\widehat{\mathbb{C}}$. A map

$$h(z, \lambda) : E \times \Delta \to \widehat{\mathbb{C}}$$

is called a holomorphic motion of E over Δ if

i) $h(z, 0) = z$ for all $z \in E$;

ii) for any fixed $\lambda \in \Delta$, $h_\lambda(\cdot) = h(\cdot, \lambda) : E \to \widehat{\mathbb{C}}$ is injective;

iii) for any fixed $z \in E$, $h^z(\cdot) = h(z, \cdot) : \Delta \to \widehat{\mathbb{C}}$ is holomorphic.

λ-Lemma and the extension problem

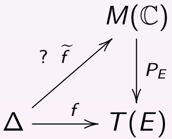
Lemma

1) *Any holomorphic motion $h(z, \lambda)$ of E over Δ can be extended to a holomorphic motion $\overline{h}(z, \lambda)$ of the closure $\overline{E}$ over Δ.*
2) *The map $\overline{h}(z, \lambda) : \overline{E} \times \Delta \to \widehat{\mathbb{C}}$ is a continuous map.*
3) *For any given $\lambda \in \Delta$, the map $\overline{h}_\lambda(\cdot) = \overline{h}(\cdot, \lambda) : \overline{E} \to \widehat{\mathbb{C}}$ has some "quasiconformal property".*

The extension problem: can we extend the holomorphic motion $h(z, \lambda)$ further to a holomorphic motion $H(z, \lambda)$ of $\widehat{\mathbb{C}}$ over Δ?

The Lifting problem in Teichmüller theory

Immediately after holomorphic motions were defined, Bers and Royden, 1986, realized in their paper that the extension problem is important in the study of Teichmüller theory because of the lifting problem in Teichmüller theory as follows:



They used to call the lifting problem as one of the most important problems in Teichmüller theory.

Explanation of the lifting problem

$M(\mathbb{C}) = \{\mu \in L^\infty(\mathbb{C}) \mid \|\mu\|_\infty < 1\}$ is the unit ball (called the space of Beltrami coefficients) in $L^\infty(\mathbb{C})$.

When E contains only finite number of points, $T(E) = Teich(\Omega)$ is the Teichmüller space of the Riemann surface $\Omega = \widehat{\mathbb{C}} \setminus E$ and a domain in $\mathbb{C}^{\#(E)-3}$. In this case, the lifting problem is a problem in several complex variables. We will define $T(E)$ for any closed subset E of the Riemann sphere $\widehat{\mathbb{C}}$.

The map $P_E : M(\mathbb{C}) \to T(E)$ is the holomorphic projection.

The lifting problem: Given any basepoint-preserving holomorphic map $f : \Delta \to T(E)$, can we find a basepoint-preserving holomorphic map $\widetilde{f} : \Delta \to M(\mathbb{C})$ such that $P_E \circ \widetilde{f} = f$?

The measurable Riemann mapping theorem

Theorem

For any $\mu \in M(\mathbb{C})$, the Beltrami equation

$$w_{\overline{z}} = \mu w_z$$

always has a solution w which is a K-quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ for $K = (1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty)$. Moreover, if we consider the normalized solution w^μ fixing 0, 1, ∞, then w^μ is unique and depends on μ holomorphically.

Gauss 1822, isothermal coordinate on surfaces.
Morrey 1932, quasi-linear elliptic partial differential equation.
Bojarski 1955, Beltrami equations.
etc. ....
Ahlfors-Bers, 1960, a famous paper.

Teichmüller space $T(E)$ of a closed subset E

Earle and his students Lieb and Mitra have studied the Teichmüller space of a closed subset $0, 1, \infty \in E$ of the Riemann sphere $\widehat{\mathbb{C}}$ carefully in their papers (Lieb, 1990 and Earle and Mitra, 2000):

We say $\mu, \nu \in M(\mathbb{C})$ are E-equivalent if $(w^{\nu})^{-1} \circ w^{\mu}$ is isotopic to the identity rel E. The space of all E-equivalence classes

$$T(E) = \{[\mu]_E \mid \mu \in M(\mathbb{C})\}$$

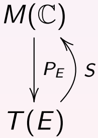
is called the Teichmüller space of the closed subset E. It is a simply connected contractible complex Banach manifold such that the projection

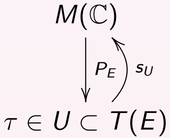
$$P_E(\mu) = [\mu]_E : M(\mathbb{C}) \to T(E)$$

is a holomorphic split submersion.

Holomorphic split submersion

For the projection $P_E : M(\mathbb{C}) \to T(E)$, there is no global holomorphic section as long as $\#(E) > 4$. Instead, there is a continuous (real analytic) global section $S : T(E) \to M(\mathbb{C})$ such that $P_E \circ S = Id$ (based on the barycentric (or Douady-Earle) extension). However, there is a local holomorphic section on a neighborhood of any point (based one Ahlfors-Weill extension), that is, for any $\tau \in T(E)$, there is an open set $\tau \in U \subset T(E)$ and a holomorphic map $s_U : U \to M(\mathbb{C})$ such that $P_E \circ s_U = Id$.





Not perfect full extension

Using the holomorphic split submersion property, Bers and Royden, 1986, proved the following theorem.

Theorem

Any holomorphic motion $h(z,\lambda)$ of E over Δ can be extended to a holomorphic motion $\widetilde{h}(z,\lambda)$ of $\widehat{\mathbb{C}}$ over $\Delta_{1/3} = \{\lambda \mid |\lambda| < 1/3\}$.

Sullivan and Thurston, 1986, proved a similar result.

Theorem

Any holomorphic motion $h(z,\lambda)$ of E over Δ can be extended to a holomorphic motion $\widetilde{h}(z,\lambda)$ of $\widehat{\mathbb{C}}$ over $\Delta_r = \{\lambda \mid |\lambda| < r\}$ for some $0 < r < 1$.

Slodkowski's Theorem

A full extension is proved by Slodkowski, 1991, in his paper by using several complex variables.

Theorem

Any holomorphic motion $h(z, \lambda)$ of a subset E in the Riemann sphere over Δ can be extended to a holomorphic motion $H(z, \lambda)$ of $\widehat{\mathbb{C}}$ over Δ.

Several people tried to prove this theorem by using different methods. However, according to Bers and Royden, 1986, they wanted to have a proof based on the study of generalized Beltrami equations since they believed that Slodkowski's Theorem should be treated as a converse part of the measurable Riemann mapping theorem.

General definition of a holomorphic motion

Suppose V is a connected complex manifold with a basepoint t_0 and E is a subset of the Riemann sphere $\widehat{\mathbb{C}}$. A map

$$h(z,t) : E \times V \to \widehat{\mathbb{C}}$$

is called a holomorphic motion of E over V if

i) $h(z, t_0) = z$ for all $z \in E$;

ii) for any fixed $t \in V$, $h_t(\cdot) = h(\cdot, t) : E \to \widehat{\mathbb{C}}$ is injective;

iii) for any fixed $z \in E$, $h^z(\cdot) = h(z, \cdot) : V \to \widehat{\mathbb{C}}$ is holomorphic.

λ-Lemma for the general definition

Similarly, we have that

Lemma

1) *Any holomorphic motion $h(z,t)$ of E over V can be extended to a holomorphic motion $\overline{h}(z,t)$ of the closure $\overline{E}$ over V.*
2) *The map $\overline{h}(z,t) : \overline{E} \times V \to \widehat{\mathbb{C}}$ is a continuous map.*
3) *For any given $t \in V$, the map $\overline{h}_t(\cdot) = \overline{h}(\cdot, t) : \overline{E} \to \widehat{\mathbb{C}}$ has some "quasiconformal property".*

Thus without loss of generality, we always assume that E is closed and contains $\{0, 1, \infty\}$ and that $h(0,t) = 0$, $h(1,t) = 1$, and $h(\infty, t) = \infty$ for all $t \in V$. We call such a holomorphic motion a normalized holomorphic motion.

Universal holomorphic motion

Consider the Teichmüller space $T(E)$ as V with the basepoint $t_0 = [0]_E$ and define

$$\Psi_E(z,t) = w^{\mu}(z) : E \times T(E) \to \widehat{\mathbb{C}}, \quad t = P_E(\mu),\ \mu \in M(\mathbb{C}).$$

It is a holomorphic motion of E over $T(E)$. , Mitra, 2000, proved the following theorem.

Theorem

Suppose V is a simply connected complex Banach manifold. Then any holomorphic motion $h(z,t) : E \times V \to \widehat{\mathbb{C}}$ can be treated as the pullback $f^*(\Psi_E)(t,z)$ of Ψ_E for a unique basepoint-preserving holomorphic map $f : V \to T(E)$, that is,

$$h(z,t) = f^*(\Psi_E)(z,t) := \Psi(z, f(t)).$$

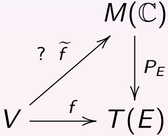
Thus we call Ψ_E a universal holomorphic motion.

The extension problem and the lifting problem

The extension problem: given a holomorphic motion $h(z,t)$ of E over V, under what condition, can we extend it to a holomorphic motion $H(z,t)$ of $\widehat{\mathbb{C}}$ over V?

If the answer is yes, we call h fully extendable.

The lifting problem: given any basepoint-preserving holomorphic map $f : V \to T(E)$, under what condition, we can find a basepoint-preserving holomorphic map $\widetilde{f} : V \to M(\mathbb{C})$ such that $P_E \circ \widetilde{f} = f$?



From the universal holomorphic motion property, when V is a simply connected complex Banach manifold, the lifting problem and the extension problem are equivalent.

Higher-dimensional simply connected case

There is a holomorphic motion of E with $\#(E) > 4$ over a simply connected higher-dimensional complex manifold V such that h is not fully extendable.

Earle, 1969

Hubbard, 1976

J-Mitra, 2006.

Douady and Earle's conterexample in one-dimensional case

Let $E = \{0, 1, \infty, t_0\}$ be a four-point set and $\mathbb{C}_{0,1} = \mathbb{C} \setminus \{0, 1\}$ be the thrice-punctured sphere with a basepoint t_0,

$$h_1(z,t) = \begin{cases} z & \text{if } z = 0, 1, \infty \text{ and } t \in \mathbb{C}_{0,1}; \\ t & \text{if } z = t_0 \text{ and } t \in \mathbb{C}_{0,1}. \end{cases}$$

Earle modified this example to get a counterexample h_2 of

$E = \{0, 1, \infty, t_0\}$ over an annulus A with a basepoint t_0.
Earle, 1997, proved that h_1 and h_2 are not fully extendable by using some maximal property.

A counterexample from our group in one-dimensional case

I used to ask my students to construct a different type counterexample from the ones from Douady and Earle. One of my students, Beck, 2003, (unpublished), returned me a counterexample of a holomorphic motion of a five-point set over the punctured disk.

Let $E = \{0, 1, \infty, a = -t_0 + 2i, b = t_0 + 2i\}$ and let $\Delta^* = \{t \in \mathbb{C} \mid 0 < |t| < 1\}$ be the the punctured unit disk with a basepoint t_0. Define

$$h_3(z,t) = \begin{cases} z & \text{if } z = 0, 1, \infty \text{ and } t \in \Delta^*; \\ -t + 2i & \text{if } z = a \text{ and } t \in \Delta^*; \\ t + 2i & \text{if } z = b \text{ and } t \in \Delta^*. \end{cases}$$

It is a holomorphic motion of E over Δ^* and not fully extendable.

Winding Number

Suppose $h(z,t)$ is a holomorphic motion of E over a hyperbolic Riemann surface X with a basepoint t_0. Suppose $\alpha(\theta) : [0,1] \to X$ is a simple closed curve and $z_1 \neq z_2 \in E$ are a pair of points. Then $\delta(\alpha, z_1, z_2) = h(\alpha, z_1) - h(\alpha, z_2)$ is a closed curve and

$$\eta(\alpha, z_1, z_2) = \frac{1}{2\pi} \oint_{\alpha} d \arg \delta(\cdot, z_1, z_2)$$

is its winding number with respect to 0.

The zero winding number condition: if $\eta(\alpha, z_1, z_2) = 0$ for all α and all pairs $z_1 \neq z_2 \in E$.

We know in Chirka, 2004 and Beck-J-Mitra-Shiga, 2012, that a holomorphic motion h is fully extendable, then it must satisfy the zero winding number condition. All the counterexamples h_1, h_2, and h_3 does not satisfy the zero winding number condition. Thus they are not fully extendable.

Chirka's work

Chirka, 2004, asserted that a holomorphic motion $h(z, t)$ of E over a hyperbolic Riemann surface is fully extendable if and only if it satisfies the zero winding number condition.

Soon after the paper was published, many people felt that the sufficient part may not be true but need a counterexample. However, Chirka's work opens a new way to understand the extension problem and brings the study of the extension problem back to Bers and Royden's original idea by using generalized Beltrami equations,

$$w_{\overline{z}} = \mu w_z + \psi.$$

Moreover, Chirka's work gives a new and comprehensive proof of Slodkowski's theorem.

My example

Let $E = \{0, 1, \infty, t_0\}$ be a four-point set and $X = \{t_0/R < |z| < t_0 R\}$ be an annulus, where $R > 1$ and $t_0 > 0$ and both $R - 1$ and $z_0 = 1/t_0$ are small numbers, with a basepoint t_0. Define

$$h_4(t, z) = \begin{cases} z & \text{if } z = 0, 1, \infty \text{ and } t \in X; \\ t_0 \left(z_0 t \frac{z_0 t - a_n}{1 - a_n z_0 t} \right)^n & \text{if } z = t_0 \text{ and } t \in X. \end{cases} \qquad (1)$$

J, 2012 (unpublished), showed that h_4 is a holomorphic motion of E over X satisfying the zero winding number condition and fully extendable.

Trace-Monodomy

Suppose $h(z,t)$ is a normalized holomorphic motion of $0,1,\infty \in E$ over a hyperbolic Riemann surface X with a basepoint t_0. Let $\pi_1(X,t_0)$ denote the fundamental group of X. Then for any $z \neq 0,1,\infty \in E$, $h^z(\cdot) = h(\cdot, z) : X \to \mathbb{C}_{0,1}$ is a holomorphic map. It induces a homomorphism $\rho_{h,z} : \pi_1(X,t_0) \to \pi_1(\mathbb{C}_{0,1})$.

The trivial trace-monodomy condition: if $\rho_{h,z}$ is trivial for all $z \neq 0,1,\infty \in E$.

Beck-J-Mitra-Shiga, 2012, proved that if a holomorphic motion h is fully extendable, then it must satisfy the trivial trace-monodomy condition. The counterexamples h_1 and h_2 do not satisfy the trivial trace-monodomy condition. The counterexample h_3 and the example h_4 satisfy the trivial trace-monodomy condition. Beck-J-Mitra-Shiga, 2012, proved the Chirka's assertion but only for any four-point sets E.

Theorem

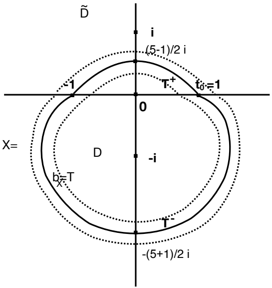
A normalized holomorphic motion h of a four-point set E over any hyperbolic Riemann surface X with a basepoint t_0 is fully extendable if and only if it satisfies the trivial trace-monodromy condition (or the zero winding number condition).

My counterexample

In, J, 2020, I constructed an explicit counterexample to Chirka's assertion. Let $E = \{0, 1, 2, 4, \infty\}$ be a five-point set. Let X be an annulus such that $-2, 0, 1/2, 1/3, i, -i \notin X$. Define

$$\phi(z,t) = \begin{cases} z & \text{if } z = 0, 1, \infty \text{ and } t \in X; \\ -\frac{1}{t} + 3 & \text{if } z = 2 \text{ and } t \in X; \\ t + 3 & \text{if } z = 4 \text{ and } t \in X. \end{cases} \tag{2}$$

Then $h(z,t) : E \times X \to \widehat{\mathbb{C}}$ is a holomorphic motion satisfying the zero winding number condition but not fully extendable.



Monodromy

The monodromy associated with a holomorphic motion $h(z,t)$ of E over a hyperbolic Riemann surface X with a basepoint t_0 is defined in, Beck-J-Mitra-Shiga, 2012 as follows:
Suppose $\pi : \Delta \to X$, $\pi(0) = t_0$, is the holomorphic universal cover with the group of deck transformations Γ. Consider the pullback holomorphic motion $H = \pi^*(h)$ of E over Δ. Since Δ is simply connected, there exists a basepoint preserving holomorphic map $f : \Delta \to T(E)$ such that $H = f^*(\Psi_E)$. For any $c \in \pi_1(X, t_0)$, let β be the representation of c in Γ. Then the normalized quasiconformal homeomorphism w^μ for any μ such that $P_E(\mu) = (f \circ \beta)(0)$ fixes each point in E. Thus it is a quasiconformal self-map of the hyperbolic Riemann surface $\widehat{\mathbb{C}} \setminus E$. Therefore, it represents a mapping class $[w^\mu]$ of $\widehat{\mathbb{C}} \setminus E$.

The trivial monodromy condition

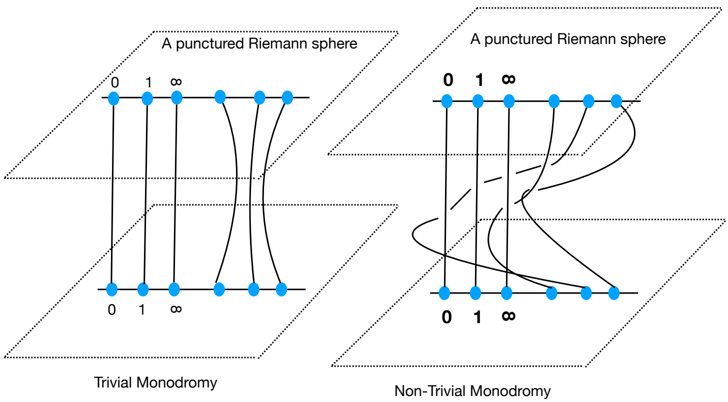
When E' contains n points, we use $\mathrm{Mod}(0, n)$ to denote the mapping class group of the n-times punctured sphere $\widehat{\mathbb{C}} \setminus E'$. Then we have a homomorphism $\rho_{E'}(c) = [w^{\mu}] : \pi_1(X, t_0) \to \mathrm{Mod}(0, n)$.
The trivial monodromy condition: if $\rho_{E'}$ is trivial for any finite subset $\{0, 1, \infty\} \subset E' \subset E$.

Beck-J-Mitra-Shiga, 2012, proved that the trivial monodromy condition is a necessary condition for the full extendibility as follows.

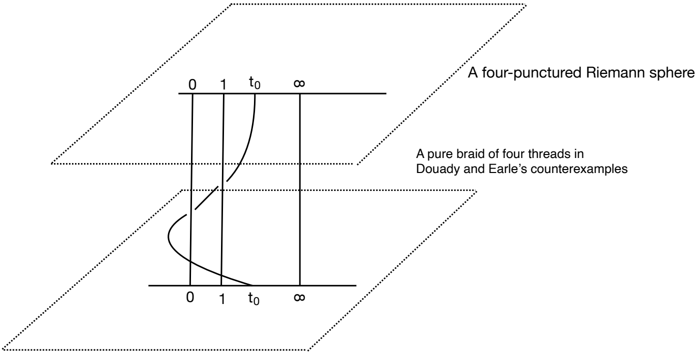
Theorem

If a holomorphic motion $h(z, t)$ of E over a hyperbolic Riemann surface X is fully extendable, then it must satisfy the trivial monodromy condition (as well as the zero winding number condition and the trivial trace-monodromy condition).

Pictures of trivial and non-trivial monodromy

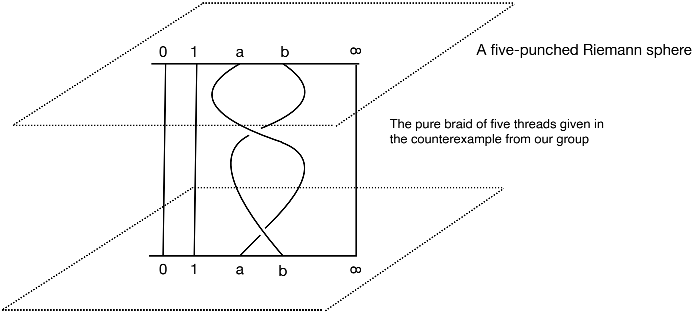


Monodromy of h_1 and h_2



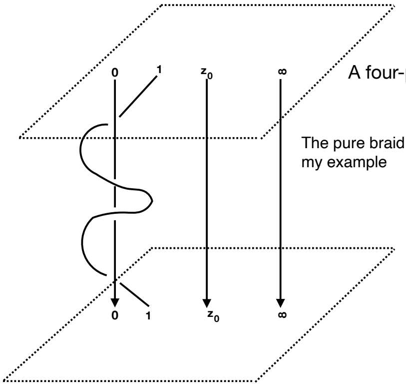
This example does not satisfy the zero winding number condition, the trivial trace-monodromy condition, and the trivial monodromy condition. Thus it is not fully extendable

Monodromy of h_3



This example satisfies the trivial trace-monodromy condition and does not satisfy the zero winding number condition and the trivial monodromy condition. Thus it is not fully extendable.

Monodromy of h_4

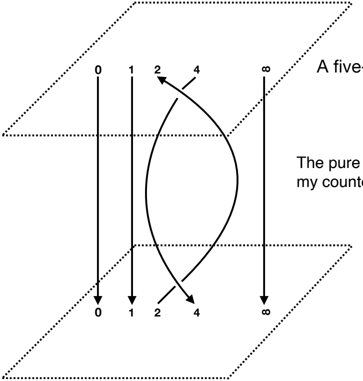


A four-punched Riemann sphere

The pure braid of four threads given in my example

This example satisfies the zero winding number condition,
the trivial trace-monodromy condition,
and the trivial monodromy condition.
It is fully extendable

Monodromy of h_5



A five-punctured Riemann sphere

The pure braid of five threads given in my counterexample

This example satisfies the zero winding number condition and the trivial trace-monodromy condition but does not satisfy the trivial monodromy condition. Thus it is not fully extendable

Holomorphic map problem

Holomorphic map problem: In general, for a connected complex Banach manifold V with a basepoint and for a given holomorphic motion h of E over V, under what condition, can we find a basepoint-preserving holomorphic map $f : V \to T(E)$ such that $f^*(\Psi_E) = h$?

The holomorphic map problem and the lifting problem together are equivalent to the extension problem.

Sufficiency for the holomorphic map problem

J-Mitra, 2018, proved that the trivial monodromy condition is indeed a sufficient condition for the holomorphic map problem.

Theorem

Suppose h is a holomorphic motion of E over a hyperbolic Riemann surface X with a basepoint t_0 satisfying the trivial monodromy condition. Then for any finite subset $\{0, 1, \infty\} \subset E' \subset E$, we have a basepoint-preserving holomorphic map $f_{E'} : X \to T(E')$ such that $f_{E'}^*(\Psi_{E'}) = h|X \times E'$.

Sufficiency for the lifting problem

J-Mitra-Wang, 2009 and J-Mitra, 2018 combined, proved that the trivial monodromy condition is indeed a sufficient condition for the lifting problem.

Theorem

Suppose h is a holomorphic motion of E over a hyperbolic Riemann surface X with a basepoint t_0 satisfying the trivial monodromy condition. Then for any finite subset $\{0, 1, \infty\} \subset E' \subset E$, suppose $f_{E'} : X \to T(E')$ is the basepoint preserving holomorphic map in the previous page. Then we have a basepoint-preserving holomorphic map $\widetilde{f}_{E'} : X \to M(\mathbb{C})$ such that $P_{E'} \circ \widetilde{f}_{E'} = f_{E'}$.

Finally, J-Mitra, 2018, completed a research project we been worked on for quite long times by following Bers and Royden's original idea by using generalized Beltrami equations to study the extension problem, thanks to Chirka's work.

Theorem

Suppose $h(z, t)$ is a holomorphic motion of a subset E in the Riemann sphere $\widehat{\mathbb{C}}$ over a hyperbolic Riemann surface X with a basepoint t_0. Then it is fully extendable if and only if it satisfies the trivial monodromy condition.

Guiding Isotopy

Gardiner-J-Wang, 2015, has used another criterion called guiding isotopy to study the full extendibility: A holomorphic motion $h(z,t)$ of E over a hyperbolic Riemann surface X with basepoint t_0 is said to have guiding isotopy if for any finite subset $E' \subset E$, $h' = h|X \times E'$ can be extended to a continuous motion $H'(z,t)$ of the Riemann sphere $\widehat{\mathbb{C}}$ over X, that is,

1) $H'(z,t_0) = z$ for all $z \in \widehat{\mathbb{C}}$ and 2) for each $t \in X$ and
2) $H'_t(z) = H'(z,t) : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a homeomorphism.

Continuous motion and quasiconformal motion

In the original definition of guiding isotopy in Gardiner-J-Wang, 2015, we require $h' = h|X \times E'$ can be extended to a quasiconformal motion which is a concept introduced in Sullivan and Thurston, 1986 as follows:
Suppose W is a connected Hausdorff space with a basepoint w_0. A map $\phi(z, w) : E \times W \to \mathbb{C}$ is called a quasiconformal motion if

a) $\phi(z, w_0) = z$ for all $z \in E$;
b) for each $w \in W$, the map $\phi(\cdot, w) : E \to \mathbb{C}$ is injective;
c) given any $w \in W$ and any $\epsilon > 0$, there is a neighborhood U_w about w such that for any quadruplet $a, b, c, d \in E$,

$$\rho_{0,1}(\phi_x(a, b, c, d), \phi_y(a, b, c, d)) < \epsilon, \ \ \forall\, x, y \in U_w,$$

where $\rho_{0,1}$ is the hyperbolic metric on the thrice punctured sphere and

$$\phi_x(a, b, c, d) = \frac{(\phi_x(a) - \phi_x(c))(\phi_x(b) - \phi_x(d))}{(\phi_x(a) - \phi_x(d))(\phi_x(b) - \phi_x(c))}$$

is the cross-ratio.

Equivalent statements

Mitra, 2007 showed the following statements are equivalent for a holomorphic motion h of E over a connected complex Banach manifold V: (1) it can be extended to a continuous motion of $\widehat{\mathbb{C}}$ over V; (2) it can be extended to a quasiconformal motion of $\widehat{\mathbb{C}}$ over V; and (3) there exits a basepoint preserving holomorphic map $f : V \to T(E)$ such that $f^*(\Psi_E) = h$. Thus we can replace "quasiconformal motion" by "continuous motion" in the definition of guiding isotopy in Gardiner-J-Wang, 2015.

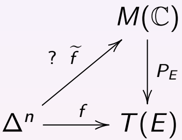
Guiding Isotopy and full extendibility

From Beck-J-Mitra-Shiga, 2012, we know that guiding isotopy implies the trivial monodromy condition. Thus the guiding isotopy implies the full extendibility from Gardiner-J-Wang, 2015 and J-Mitra, 2018.

Final conclusion

In the holomorphic motion theory, the zero winding number condition, the trivial trace-monodromy condition, the trivial monodromy condition, and guiding isotopy are all necessary conditions for the full extendibility. However, only the trivial monodromy condition and guiding isotopy conditions are, indeed, sufficient conditions for the full extendibility.

Next research problem



Here $n \geq 2$.

End

Thanks for your Listening !