Tame Quasiconformal Motions and Teichmüller Spaces

Yunping Jiang

City University of New York
NSF

A talk given in
Topology/Geometry Seminar
The Mathematics Department
Rutgers, The State University of New Jersey, New Brunswick
May 7, 2019, 3:30pm-4:30pm

Riemann Mapping Theorem

Suppose D is a non-empty simply connected open subset of the complex plane $\mathbb{C}$, but $D \neq \mathbb{C}$. Then there is a biholomorphic map

$$f : D \to \Delta = \{z \in \mathbb{C} \mid |z| < 1\}.$$

Moreover, for a given $z_0 \in D$, f is unique provided $f(z_0) = 0$ and $f'(z_0) > 0$.

Riemann 1851, Ph.D thesis;
Caráthèodory 1912, a proof.

Uniformization Theorem

Suppose S is a simply connected Riemann surface. Then S is biholomorphic to one of the following:

$$\Delta \ (\textit{hyperbolic}); \ \ \mathbb{C} \ (\textit{parabolic}); \ \ \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \ (\textit{spherical}).$$

Klein 1883;
Poincaré 1882;
Koebe 1907 and Poincaré 1907, a proof;
Abikoff 1981 survey, AMS monthly, 88, 574-592.

Measurable Riemann Mapping Theorem: Backgroud

Or called a variable metric Riemann mapping theorem.
Consider $\mathbb{R}^2 = \{(x, y)\}$ and a Riemannian metric

$$g(x, y) = E(x, y)dx^2 + 2F(x, y)dxdy + G(x, y)dy^2$$

with $E > 0$, $G > 0$, and $EG - F^2 > 0$.

Let $z = x + iy$ and $\overline{z} = x - iy$. Then

$$g(z) = \gamma(z)|dz + \mu(z)d\overline{z}|^2$$

with

$$\gamma = \frac{1}{4}(E + G - 2\sqrt{EG - F^2}), \quad \mu = \frac{E - G + 2iF}{4\gamma}.$$

Measurable Riemann Mapping Theorem: Development

Question

Does the Beltrami equation

$$w_{\overline{z}} = \mu w_z$$

has a solution?

Gauss 1822, isothermal coordinate on surfaces.

Morrey 1932, quasi-linear elliptic partial differential equation.

Ahlfors-Bers, 1960, a famous paper.

Bojarski 1955,
etc. ....

Let

$$M(\mathbb{C}) = \{\mu \in L^\infty(\mathbb{C}) \mid \|\mu\|_\infty < 1\}$$

For any $\mu \in M(\mathbb{C})$, the Beltrami equation always has a solution w which is a K-quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ for $K = (1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty)$. Moreover, if we consider the normalized solution w^μ fixing 0, 1, ∞, then w^μ is unique and depends on μ holomorphically.

Quasiconformal Homeomorphism

Analytic definition: A $W^{1,2}_{loc}$ (first-order distribution partial derivatives in L^2_{loc}) map satisfies the Beltrami equation as a weak solution.

Geometric definition: A map increases the modulus of any quadrilateral at most by K.

Grötzsch 1928, Ahlfors 1935 formally introduced

Teichmüller Equivalence rel to a Closed Subset

Let E be a closed subset of $\widehat{\mathbb{C}}$. Suppose $0, 1, \infty \in E$. We say $\mu, \nu \in M(\mathbb{C})$ are E-equivalent, denote as $\mu \sim_E \nu$ if $(w^\nu)^{-1} \circ w^\mu$ is homotopic to the identity rel to E, that is, there is a continuous map $H(t, z) : [0, 1] \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

1) $H(0, z) = z$;
2) $H(1, z) = (w^\nu)^{-1} \circ w^\mu(z)$ for all $z \in \widehat{\mathbb{C}}$; and
3) $H(t, z) = z$ for all $0 \leq t \leq 1$ and all $z \in E$.

Teichmüller Space of a Closed Subset

The space of all equivalence classes

$$T(E) = \{[\mu] \mid \mu \in M(\mathbb{C})\}$$

is called the Teichmüller space of the closed subset E.

Teichmüller Metric

For any $\mu, \nu \in M(\mathbb{C})$, let

$$d_0(\mu,\nu) = \tanh^{-1} \left\| \frac{\mu-\nu}{1-\overline{\mu}\nu} \right\|_\infty = \frac{1}{2} \log \frac{1 + \left\| \frac{\mu-\nu}{1-\overline{\mu}\nu} \right\|_\infty}{1 - \left\| \frac{\mu-\nu}{1-\overline{\mu}\nu} \right\|_\infty}$$

The Teichmüller metric on $T(E)$ is, for any $\alpha, \beta \in T(E)$,

$$d_T(\alpha,\beta) = \inf\{ d_0(\mu,\nu) \mid \mu \in \alpha, \ \ \nu \in \beta \}$$

Product Teichmüller Space

The complement $\widehat{\mathbb{C}} \setminus E = \cup_i \Omega_i$. Each Ω_i is a Riemann surface. So we have the classical Teichmüller space $Teich(\Omega_i)$. We then have a product Teichmüller space

$$\prod_i Teich(\Omega_i) = \{(\tau_i) \mid \tau_i \in Teich(\Omega_i),\ \sup_i d_{\Omega_i,T}(0_i, \tau_i) < \infty\}$$

Isomorprhism

There is a biholomorphic map between the Teichmüller space $T(E)$ and the product Teichmüller space $\prod_i Teich(\Omega_i) \times M(E)$, that is,

$$T(E) \simeq \prod_i Teich(\Omega_i) \times M(E).$$

Lieb PhD Thesis 1990; also see Earle-Mitra 2000 or J-Mitra 2012 for a proof.

Holomorphic Projection and Local Holomorphic Section

The map $P_E(\mu) = [\mu]$ is a holomorphic split submersion, that is, for any $\tau \in T(E)$, there is a neighborhood U about τ and a holomorphic section $s_{\tau,U} : U \to M(\mathbb{C})$ such that $P_E \circ s_{\tau,U} = Id$.

$$\begin{array}{c} M(\mathbb{C}) \\ P_E \downarrow \; \uparrow s_{\tau,U} \\ T(E) \end{array}$$

Global Continuous Section

The map $P_E(\mu) = [\mu]$ has a global continuous section, that is, there is a continuous section $S : T(E) \to M(\mathbb{C})$ such that $P_E \circ S = Id$ (from Douady-Earle 1986 paper about the barycentric extension of a quasisymmertic homeomorphism of the circle).

$$\begin{array}{c} M(\mathbb{C}) \\ P_E \big\downarrow \; \big\uparrow S \\ T(E) \end{array}$$

This implies that $T(E)$ is contractible.

If $\dim T(E) \geq 2$, then P_E can not have a global holomorphic section (Earle 1969).

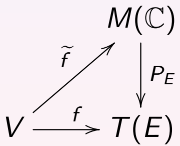
Geometry

Complex Geometry: all basepoint preserving holomorphic maps $f: V \to T(E)$ where V is a connected complex Banach manifold with a basepoint t_0. In particular, $V = \Delta$ with metric $d\rho = |dz|/(1-|z|^2)$.

Real Geomtry: all basepoint preserving continuous maps $f: W \to T(E)$ where W is a connected Hausdorff space with a based point t_0. In particular, $W = I = [0,1]$ is the unit interval.

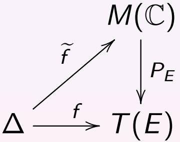
Problem (Lifting Problem)

For a basepoint preserving holomorphic map $f : V \to T(E)$, *can one find a basepoint preserving holomorphic map* $\widetilde{f} : V \to M(\mathbb{C})$ *such that* $P_E \circ \widetilde{f} = f$*?*



Lifting Theorem for Δ

For $V = \Delta$ with the basepoint 0, the answer is affirmative, that is, for any basepoint preserving holomorphic map $f : \Delta \to T(E)$, one can find a basepoint preserving holomorphic map $\widetilde{f} : \Delta \to M(\mathbb{C})$ such that $P_E \circ \widetilde{f} = f$,



Reference: Yunping Jiang, Sudeb Mitra, and Zhe Wang, Liftings of holomorphic maps into Teichmüller spaces. Kodai Mathematical Journal, 32 (2009), No. 3, 544-560.

One application of this lifting theorem is an easy proof of Teichmüller's metric = Kobayahi's Metric (Royden, 1971; Gardiner, 1984).

Holomorphic Motions

Suppose V is a connected complex Banach manifold with basepoint t_0. A map $h(t, z) : V \times E \to \widehat{\mathbb{C}}$ is called a holomorphic motion if

i) $h(t_0, z) = z$ for all $z \in E$;

ii) for any fixed $t \in V$, $h(t, \cdot) : E \to \widehat{\mathbb{C}}$ is injective;

iii) for any fixed $z \in E$, $h(\cdot, z) : V \to \widehat{\mathbb{C}}$ is holomorphic.

We can normalized it by assuming $h(t, 0) = 0$, $h(t, 1) = 1$, and $h(t, \infty) = \infty$ for all $t \in V$.

Example: The map $\Psi_E(t, z) = w^\mu(z) : T(E) \times E \to \widehat{\mathbb{C}}$ is a holomorphic motion.

Universal Property for Holomorphic Motions

The holomorphic motion $\Psi_E(t,z) = w^{\mu}(z) : T(E) \times E \to \widehat{\mathbb{C}}$ is universal for holomorphic motions in the meaning that for any holomorphic motion $h(t,z) : V \times E \to \widehat{\mathbb{C}}$, where V is a simply connected Banach complex manifold with a basepoint, there is a unique basepoint preserving map $f : V \to T(E)$ such that $f^*(\Psi_E) = h$.

Earle 1988, Mitra PhD Thesis 1994 (2007), see also J-Mitra 2012.

Extension Problem

We say $H(t,z): V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a holomorphic motion extension of a holomorphic motion $h(t,z): V \times E \to \widehat{\mathbb{C}}$ if $H|(V \times E) = h$.

Problem (Extension Problem)

For a given holomorphic motion $h(t,z): V \times E \to \widehat{\mathbb{C}}$, can we extend it to a holomorphic motion $H(t,z): V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$?

Holomorphic Lifting Problem and the Extension Problem

Due to the universal property, the extension prooblem is equivalent to the holomorphic lifting problem.

When $V = \Delta$, it is Slodkowski's theorem (Slodowski 1991, Bers-Royden 1986, Sullivan-Thurston 1986, Chirka 2004, also see J-Mitra-Wang 2009 for a direct proof of the lifting problem), that is, the answer to both problems is yes.

For $V = T(E)$ with $\dim T(E) \geq 2$, the answer is no (J-Mitra 2007, Earle 1968).

Quasiconformal Motion

Definition

Suppose W is a connected Hausdorff space with a basepoint t_0. A map $\phi(t, z) : W \times E \to \mathbb{C}$ is called a quasiconformal motion if

i) $\phi(t_0, z) = z$ for all $z \in E$;

ii for each $t \in W$, the map $\phi(t, \cdot) : E \to \mathbb{C}$ is injective;

iii) given any $t \in W$ and any $\epsilon > 0$, there is a neighborhood U_t about t such that for any quadruplet $a, b, c, d \in E$,

$$\rho_{0,1}(\phi_x(a, b, c, d), \phi_y(a, b, c, d)) < \epsilon, \ \ \forall \, x, y \in U_t.$$

Sullivan-Thurston, 1986, Acta Math

Notation

In the above definition, $\rho_{0,1}$ is the hyperbolic metric on the thrice punctured sphere $\mathbb{C} \setminus \{0, 1\}$ and

$$\phi_x(a, b, c, d) = \frac{(\phi_x(a) - \phi_x(c))(\phi_x(b) - \phi_x(d))}{(\phi_x(a) - \phi_x(d))(\phi_x(b) - \phi_x(c))}$$

is the cross-ratio of four points $\phi_x(z) = \phi(x, z)$ for $z = a, b, c, d$.

Universal Problem for Quasiconformal Motions

When we study the real geometry on $T(E)$, the following problem is important.

Problem (Universal Problem)

Whether $\Psi_E(t,z) : T(E) \times E \to \widehat{\mathbb{C}}$ *is a universal for quasiconformal motion? That is, for any quasiconformal motion,* $\phi(t,z) : W \times E \to \widehat{\mathbb{C}}$, *where* W *is a simply connected Hausdorff space with a basepoint* t_0, *can we find a basepoint preserving continuous map* $f : W \to T(E)$ *such that* $f^*(\Psi_E) = \phi$*?*

Extension Problem for Quasiconformal Motions

Problem (Extension for Quasiconformal Motions)

For a quasiconformal motion $\phi(t,z) : W \times E \to \widehat{\mathbb{C}}$, can one extend it to a quasiconformal motion $\Phi(t,z) : W \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$?

Note that for a quasiconformal motion $\Phi(t,z) : W \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and each $t \in W$, $\Phi_t(\cdot) : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasiconformal homeomorphism and $t \to \mu_{\Phi_t}$ is a continuous map from W to $M(\mathbb{C})$,

Counterexample for QC Extension

In their 1986 paper, Thurston and Sullivan asserted that the extenion problem holds when $W = [0,1]$. However, in our 2013 work, J-Mitra-Shiga-Wang, we proved the following theorem. Let $I = [0,1]$ with the basepoint 0.

Theorem (J-Mitra-Shiga-Wang, Tohoku, 2018)

There exists a closed subset E in $\widehat{\mathbb{C}}$ with $\#(E) = \infty$, and a quasiconformal motion $\phi(t,z) : I \times E \to \widehat{\mathbb{C}}$, such that ϕ CANNOT be extended to a quasiconformal motion $\Phi(t,z) : U \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ for any neighborhood U of the basepoint 0. However, it can be extended to a continuous motion of $\widehat{\mathbb{C}}$ over I.

In the theorem, I can be replaced by any path connected Hausdorff space W.

Outline of the Proof, I

We take $1 < r_1 < r_2 < \cdots < r_n < r_{n+1} < \cdots$ so that $r_{n+1}/r_n \to \infty$ as $n \to \infty$. $X = \widehat{\mathbb{C}} \setminus (\{r_n\}_{n=1}^{\infty} \cup \{\infty\})$. Let α_n be a simple closed curve in X only enclosing r_{2n} and r_{2n+1}.

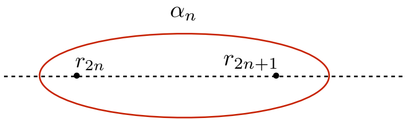


Fig. 1

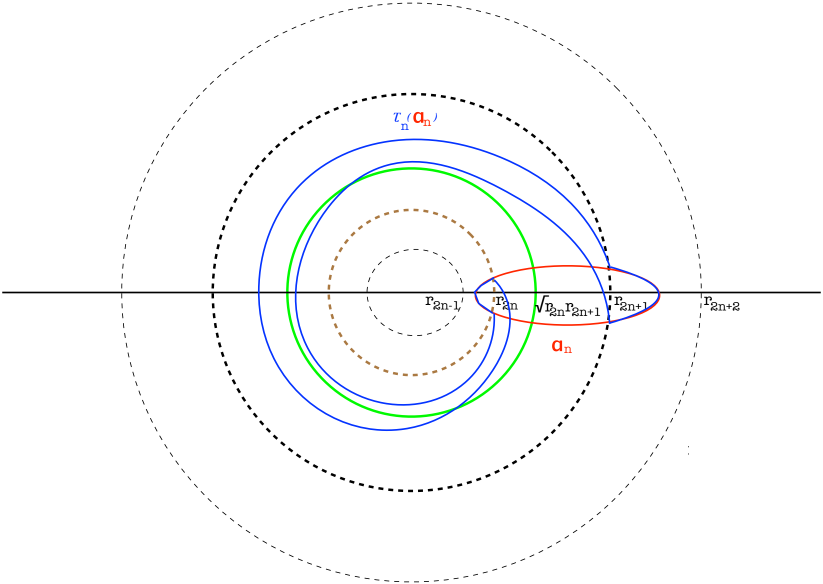
Outline of the Proof, II

Let $A_n = \{r_{2n} \leq |z| \leq r_{2n+1}\}$ and $B_n = \{r_{2n+1} \leq |z| \leq r_{2n+2}\}$.
Take $p_n \in \mathbb{N}$ so large that

$$\lim_{n\to\infty} \frac{l_X(\tau_n^{p_n}(\alpha_n))}{l_X(\alpha_n)} = \infty$$

where τ_n is the Dehn twist in A_n about its core curve and $l_X(\alpha_n)$ and $l_X(\tau_n^{p_n}(\alpha_n))$ mean the hyperbolic lengths of closed geodesics homotopic to α_n and $\tau_n^{p_n}(\alpha_n)$ in X, respectively.

Figure 2



Outline of the Proof, III

Let $E = \cup_{n=1}^{\infty} C_n \cup \{\infty\}$ with $C_n = \{|z| = r_n\}$. For each $n \in \mathbb{N}$, define $\phi_n(t,z) = z \exp\{2\pi i n(n+1)(t-(n+1)^{-1})p_n\}$ for $(t,z) \in [(n+1)^{-1}, n^{-1}] \times C_{2n}$ and $\phi_n(t,z) = z$ elsewhere. Define

$$\phi(t,z) = \lim_{n\to\infty} \phi_n \circ \cdots \circ \phi_1(t,z) : I \times E \to \widehat{\mathbb{C}}.$$

Then we have proven that

a) ϕ can be extended to a continuous motion of $\widehat{\mathbb{C}}$ over I.
b) ϕ is a quasiconformal motion of E over I.
c) ϕ can not be extended to a quasiconformal motion of $\widehat{\mathbb{C}}$ over any U.

Outline of the Proof, IV

In the proof of c), we use a result of Wolpert 1979: Let X and Y be hyperbolic surfaces and $f : X \to Y$ be a K-quasiconformal map from X to Y. Then, for any non-trivial and non-peripheral closed curve α on X,

$$\frac{1}{K} l_X(\alpha) \leq l_Y(f(\alpha)) \leq K l_X(\alpha)$$

holds, where $l_X(\alpha)$ is the hyperbolic length of the geodesic on X homotopic to α in X, $l_Y(f(\alpha))$ is the hyperbolic length of the geodesic on Y homotopic to $f(\alpha)$.

The Universal Property Fails for Quasiconformal Motions

Corollary (J-Mitra-Shiga-Wang, Tohoku, 2018)

The motion $\Psi(t,z) = w^{\mu}(z) : T(E) \times E \to \widehat{\mathbb{C}}$ *is not universal for quasiconformal motions.*

Tame Quasiconformal Motion

Definition (J-Mitra-Shiga-Wang, Tohoku, 2018)

Let W be a connected Hausdorff space with the basepoint t_0. A map $\phi(t,z): W \times E \to \widehat{\mathbb{C}}$ is called a tame quasiconformal motion if

i) $\phi(t_0, z) = z$ for all $z \in E$;

ii) for each $t \in W$, the map $\phi(t, \cdot): E \to \mathbb{C}$ is injective;

iii) given any $t \in W$, there exists a quasiconformal map $w: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and a neighborhood U_t about t with the basepoint t and a quasiconformal motion $\psi(t,z): U_t \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\phi(x,z) = \psi(x, w(z))$ for all $(x,z) \in U_t \times E$.

Universal Property for Tame Quasiconformal Motion

Theorem (J-Mitra-Shiga-Wang, Tokohu, 2018)

The motion $\Psi(t,z) = w^{\mu}(z) : T(E) \times E \to \widehat{\mathbb{C}}$ is universal for tame quasiconformal motions. That is, for any simply connected Hausdorff space W with a basepoint t_0 and any tame quasiconformal motion $\phi(t,z) : W \times E \to \widehat{\mathbb{C}}$, there exists a unique basepoint preserving continuous map $f : W \to T(E)$ such that $f^*(\Psi_E) = \phi$.

Extension for Tame Quasiconformal Motion

Corollary (J-Mitra-Shiga-Wang, Tokoku, 2018)

Let W be a simply connected Hausdorff space with a basepoint t_0, and $\phi(t,z): W \times E \to \widehat{\mathbb{C}}$ be a tame quasiconformal motion. Then, there exists a quasiconformal motion $\Phi(t,z): W \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that Φ extends ϕ.

Group Equivariance Extension for Tame Quasiconformal Motion

Let G be a group of Möbius transformations, such that the closed set E is invariant under G, that is, $g(E) = E$ for any $g \in G$. A motion $\phi : W \times E \to \widehat{\mathbb{C}}$ is called G-equivalent if for any $g \in G$ and $x \in W$, there is a Möbius transformation $\theta_x(g)$ such that $\phi(x, g(z)) = (\theta_x(g))(\phi(x, z))$.

Corollary (G-Equivalence Extension, J-Mitra-Shiga-Wang, Tokoku, 2018)

Let W be a simply connected Hausdorff space with a basepoint, and $\phi : W \times E \to \widehat{\mathbb{C}}$ be a G-equivariant tame quasiconformal motion. Then, there exists a G-equivariant quasiconformal motion $\Phi(t, z) : W \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that Φ extends ϕ.

Thanks!