Since the eigenvector $\mathbf{v}_{r+1}$ is nonzero, it follows that

$$c_{r+1} = 0 \tag{8}$$

But equations (7) and (8) contradict the fact that $c_1, c_2, \ldots, c_{r+1}$ are not all zero so the proof is complete. ◀

## Exercise Set 5.2

▶ In Exercises 1–4, show that $A$ and $B$ are not similar matrices. ◀

**1.** $A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$

**2.** $A = \begin{bmatrix} 4 & -1 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix}$

**3.** $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**4.** $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

▶ In Exercises 5–8, find a matrix $P$ that diagonalizes $A$, and check your work by computing $P^{-1}AP$. ◀

**5.** $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

**6.** $A = \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$

**7.** $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**8.** $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

**9.** Let

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

(a) Find the eigenvalues of $A$.

(b) For each eigenvalue $\lambda$, find the rank of the matrix $\lambda I - A$.

(c) Is $A$ diagonalizable? Justify your conclusion.

**10.** Follow the directions in Exercise 9 for the matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

▶ In Exercises 11–14, find the geometric and algebraic multiplicity of each eigenvalue of the matrix $A$, and determine whether $A$ is diagonalizable. If $A$ is diagonalizable, then find a matrix $P$ that diagonalizes $A$, and find $P^{-1}AP$. ◀

**11.** $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

**12.** $A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$

**13.** $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

**14.** $A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

▶ In each part of Exercises 15–16, the characteristic equation of a matrix $A$ is given. Find the size of the matrix and the possible dimensions of its eigenspaces. ◀

**15.** (a) $(\lambda - 1)(\lambda + 3)(\lambda - 5) = 0$

(b) $\lambda^2(\lambda - 1)(\lambda - 2)^3 = 0$

**16.** (a) $\lambda^3(\lambda^2 - 5\lambda - 6) = 0$

(b) $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$

▶ In Exercises 17–18, use the method of Example 6 to compute the matrix $A^{10}$. ◀

**17.** $A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$

**18.** $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

**19.** Let

$$A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}$$

Confirm that $P$ diagonalizes $A$, and then compute $A^{11}$.

**20.** Let

$$A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Confirm that $P$ diagonalizes $A$, and then compute each of the following powers of $A$.

(a) $A^{1000}$ (b) $A^{-1000}$ (c) $A^{2301}$ (d) $A^{-2301}$

**21.** Find $A^n$ if $n$ is a positive integer and

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

**22.** Show that the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are similar.

**23.** We know from Table 1 that similar matrices have the same rank. Show that the converse is false by showing that the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

have the same rank but are not similar. [*Suggestion:* If they were similar, then there would be an invertible $2 \times 2$ matrix $P$ for which $AP = PB$. Show that there is no such matrix.]

**24.** We know from Table 1 that similar matrices have the same eigenvalues. Use the method of Exercise 23 to show that the converse is false by showing that the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

have the same eigenvalues but are not similar.

**25.** If $A$, $B$, and $C$ are $n \times n$ matrices such that $A$ is similar to $B$ and $B$ is similar to $C$, do you think that $A$ must be similar to $C$? Justify your answer.

**26.** (a) Is it possible for an $n \times n$ matrix to be similar to itself? Justify your answer.

(b) What can you say about an $n \times n$ matrix that is similar to $0_{n \times n}$? Justify your answer.

(c) Is is possible for a nonsingular matrix to be similar to a singular matrix? Justify your answer.

**27.** Suppose that the characteristic polynomial of some matrix $A$ is found to be $p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3$. In each part, answer the question and explain your reasoning.

(a) What can you say about the dimensions of the eigenspaces of $A$?

(b) What can you say about the dimensions of the eigenspaces if you know that $A$ is diagonalizable?

(c) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of eigenvectors of $A$, all of which correspond to the same eigenvalue of $A$, what can you say about that eigenvalue?

**28.** Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Show that

(a) $A$ is diagonalizable if $(a - d)^2 + 4bc > 0$.

(b) $A$ is not diagonalizable if $(a - d)^2 + 4bc < 0$.

[*Hint:* See Exercise 29 of Section 5.1.]

**29.** In the case where the matrix $A$ in Exercise 28 is diagonalizable, find a matrix $P$ that diagonalizes $A$. [*Hint:* See Exercise 30 of Section 5.1.]

▶ In Exercises 30–33, find the standard matrix $A$ for the given linear operator, and determine whether that matrix is diagonalizable. If diagonalizable, find a matrix $P$ that diagonalizes $A$. ◀

**30.** $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$

**31.** $T(x_1, x_2) = (-x_2, -x_1)$

**32.** $T(x_1, x_2, x_3) = (8x_1 + 3x_2 - 4x_3, -3x_1 + x_2 + 3x_3, 4x_1 + 3x_2)$

**33.** $T(x_1, x_2, x_3) = (3x_1, x_2, x_1 - x_2)$

**34.** If $P$ is a fixed $n \times n$ matrix, then the similarity transformation

$$A \to P^{-1}AP$$

can be viewed as an operator $S_P(A) = P^{-1}AP$ on the vector space $M_{nn}$ of $n \times n$ matrices.

(a) Show that $S_P$ is a linear operator.

(b) Find the kernel of $S_P$.

(c) Find the rank of $S_P$.

## Working with Proofs

**35.** Prove that similar matrices have the same rank and nullity.

**36.** Prove that similar matrices have the same trace.

**37.** Prove that if $A$ is diagonalizable, then so is $A^k$ for every positive integer $k$.

**38.** We know from Table 1 that similar matrices, $A$ and $B$, have the same eigenvalues. However, it is not true that those eigenvalues have the same corresponding eigenvectors for the two matrices. Prove that if $B = P^{-1}AP$, and $\mathbf{v}$ is an eigenvector of $B$ corresponding to the eigenvalue $\lambda$, then $P\mathbf{v}$ is the eigenvector of $A$ corresponding to $\lambda$.

**39.** Let $A$ be an $n \times n$ matrix, and let $q(A)$ be the matrix

$$q(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I_n$$

(a) Prove that if $B = P^{-1}AP$, then $q(B) = P^{-1}q(A)P$.

(b) Prove that if $A$ is diagonalizable, then so is $q(A)$.

**40.** Prove that if $A$ is a diagonalizable matrix, then the rank of $A$ is the number of nonzero eigenvalues of $A$.

**41.** This problem will lead you through a proof of the fact that the algebraic multiplicity of an eigenvalue of an $n \times n$ matrix $A$ is greater than or equal to the geometric multiplicity. For this purpose, assume that $\lambda_0$ is an eigenvalue with geometric multiplicity $k$.

(a) Prove that there is a basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ for $R^n$ in which the first $k$ vectors of $B$ form a basis for the eigenspace corresponding to $\lambda_0$.

(b) Let $P$ be the matrix having the vectors in $B$ as columns. Prove that the product $AP$ can be expressed as

$$AP = P\begin{bmatrix} \lambda_0 I_k & X \\ 0 & Y \end{bmatrix}$$

[*Hint:* Compare the first $k$ column vectors on both sides.]

(c) Use the result in part (b) to prove that $A$ is similar to

$$C = \begin{bmatrix} \lambda_0 I_k & X \\ 0 & Y \end{bmatrix}$$

and hence that $A$ and $C$ have the same characteristic polynomial.

(d) By considering $\det(\lambda I - C)$, prove that the characteristic polynomial of $C$ (and hence $A$) contains the factor $(\lambda - \lambda_0)$ at least $k$ times, thereby proving that the algebraic multiplicity of $\lambda_0$ is greater than or equal to the geometric multiplicity $k$.

## True-False Exercises

**TF.** In parts (a)–(i) determine whether the statement is true or false, and justify your answer.

(a) An $n \times n$ matrix with fewer than $n$ distinct eigenvalues is not diagonalizable.

(b) An $n \times n$ matrix with fewer than $n$ linearly independent eigenvectors is not diagonalizable.

(c) If $A$ and $B$ are similar $n \times n$ matrices, then there exists an invertible $n \times n$ matrix $P$ such that $PA = BP$.

(d) If $A$ is diagonalizable, then there is a unique matrix $P$ such that $P^{-1}AP$ is diagonal.

(e) If $A$ is diagonalizable and invertible, then $A^{-1}$ is diagonalizable.

(f) If $A$ is diagonalizable, then $A^T$ is diagonalizable.

(g) If there is a basis for $R^n$ consisting of eigenvectors of an $n \times n$ matrix $A$, then $A$ is diagonalizable.

(h) If every eigenvalue of a matrix $A$ has algebraic multiplicity 1, then $A$ is diagonalizable.

(i) If 0 is an eigenvalue of a matrix $A$, then $A^2$ is singular.

## Working with Technology

**T1.** Generate a random $4 \times 4$ matrix $A$ and an invertible $4 \times 4$ matrix $P$ and then confirm, as stated in Table 1, that $P^{-1}AP$ and $A$ have the same

(a) determinant.

(b) rank.

(c) nullity.

(d) trace.

(e) characteristic polynomial.

(f) eigenvalues.

**T2.** (a) Use Theorem 5.2.1 to show that the following matrix is diagonalizable.

$$A = \begin{bmatrix} -13 & -60 & -60 \\ 10 & 42 & 40 \\ -5 & -20 & -18 \end{bmatrix}$$

(b) Find a matrix $P$ that diagonalizes $A$.

(c) Use the method of Example 6 to compute $A^{10}$, and check your result by computing $A^{10}$ directly.

**T3.** Use Theorem 5.2.1 to show that the following matrix is not diagonalizable.

$$A = \begin{bmatrix} -10 & 11 & -6 \\ -15 & 16 & -10 \\ -3 & 3 & -2 \end{bmatrix}$$

# 5.3 Complex Vector Spaces

Because the characteristic equation of any square matrix can have complex solutions, the notions of complex eigenvalues and eigenvectors arise naturally, even within the context of matrices with real entries. In this section we will discuss this idea and apply our results to study symmetric matrices in more detail. A review of the essentials of complex numbers appears in the back of this text.

### Review of Complex Numbers

Recall that if $z = a + bi$ is a complex number, then:

- $\text{Re}(z) = a$ and $\text{Im}(z) = b$ are called the ***real part*** of $z$ and the ***imaginary part*** of $z$, respectively,
- $|z| = \sqrt{a^2 + b^2}$ is called the ***modulus*** (or ***absolute value***) of $z$,
- $\overline{z} = a - bi$ is called the ***complex conjugate*** of $z$,