# WHEN ELLIPSES LOOK LIKE CIRCLES: THE MEASURABLE RIEMANN MAPPING THEOREM

SAEED ZAKERI AND MAHMOOD ZEINALIAN

## 1. Introduction

In 1822 Gauss studied the problem of "representing the parts of one given surface upon another given surface that the representation shall be similar, in its smallest parts, to the surface represented [**9**]." In the modern language of differential geometry, he showed that any two surfaces are locally *conformal*. An equivalent way of stating this result is to say that given any two-dimensional manifold with a Riemannian metric, one can find local coordinates around each point in which the metric looks like a real multiple of the Euclidean metric on the plane. Such coordinates are called *isothermal*. The change of isothermal coordinates are angle-preserving maps of the plane, hence holomorphic if they are orientation-preserving. This shows that any orientable surface can be equipped with a Riemann surface structure.

In late 20's the concept of quasiconformal mappings was introduced by Grötzsch. These are homeomorphisms of the plane with $L^2_{loc}$ generalized partial derivatives which map infinitesimal circles to ellipses with bounded dilatations. The theory of quasiconformal mappings and the work of Teichmüller on the moduli space of Riemann surfaces in 30's was developed by two leading mathematicians L. Ahlfors and L. Bers and their school in 50's and early 60's. This rapid developement called for a version of Gauss's theorem which would treat *families* of *measurable* Riemannian metrics on a surface as well. In 1960, using the earlier results of Calderon and Zygmund on singular integral transforms, Ahlfors and Bers proved the required version which today is known as the "Measurable Riemann Mapping Theorem" [**3**]. However, Bers realized that a version of the theorem (without dependence on parameters) had already been proved by C.B. Morrey in 1938 [**14**].

The theorem of Ahlfors-Bers-Morrey soon became the basic tool in the study of Teichmüller spaces and Kleinian groups. Later, in early

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80's, Sullivan, Shishikura, Douady and Hubbard used the theorem in *quasiconformal surgery* and the study of iteration of rational functions on the sphere. It turned out that quasiconformality is the exact level of regularity one needs in the course of a surgery. The fact that the conformal structures involved in the theorem are only measurable exhibits its extremely flexible nature as a tool for surgery. This idea led to the proof of two old conjectures in conformal dynamics [**16**],[**18**] and was the origin of the theory of polynomial-like maps [**7**].

The goal of this paper is to give a self-contained expository account of what this theorem is about, and to show a few applications.

## 2. Conformal Structures on Surfaces

Let us consider a $C^\infty$ smooth, connected, oriented surface $X$. A **conformal structure** $\sigma$ on $X$ is an equivalence class of measurable Riemannian metrics in the following sense: If a metric $g$ has the local expression

$$g(x,y) = E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2,$$

then $g$ is called measurable if $E, F$, and $G$ are (Lebesgue) measurable functions of $x, y$ such that $E > 0$, $G > 0$, and $EG - F^2 > 0$ almost everywhere. Two metrics $g_1$ and $g_2$ are said to be equivalent iff $g_1 = \gamma g_2$ for some positive measurable function $\gamma$ on $X$. Thus, the notion of the angle between two tangent vectors at almost every point is well-defined in the conformal class of a particular metric.

Now suppose in addition that $X$ is a Riemann surface, i.e., suppose that there is a complex structure $\mathcal{A}$ (a maximal $\mathbb{C}$-analytic atlas) associated to $X$. In this case it is much more useful to choose the usual complex-variable notation for our metrics. A metric $g$ can then be written as

$$g(z) = \gamma(z)|dz + \mu(z)d\bar{z}|^2, \tag{1}$$

where $z = x + iy$, $\gamma$ and $\mu$ are measurable functions of $z$, $\gamma(z) > 0$ and $|\mu(z)| < 1$ for almost every $z$ (cf. Appendix A). It is easily seen that $\mu(z)$ is not a well-defined function on $X$. However, the expression $\mu(z)d\bar{z}/dz$ is well-defined under holomorphic change of coordinates. In fact, if $z \mapsto w$ is a change of coordinates on $X$ and if $g$ in (1) has the local expression $\tilde{\gamma}(w)|dw + \tilde{\mu}(w)d\bar{w}|^2$ in $w$, then

$$\tilde{\mu}(w) = \mu(z)\frac{(dw/dz)}{\overline{(dw/dz)}}, \tag{2}$$

from which it follows that $\mu(z)d\bar{z}/dz = \tilde{\mu}(w)d\bar{w}/dw$. The expression

$$\mu = \mu(z)\frac{d\bar{z}}{dz}$$

is called the **Beltrami differential** of the metric $g$. From the definition of a conformal structure and (1) it follows that $\mu$ depends only on the conformal class of $g$.

Conversely, given any measurable Beltrami differential $\mu$, we can consider the associated conformal structure which is the conformal class of the metric $|dz+\mu(z)d\bar{z}|^2$. As a result, on a Riemann surface there is a one-to-one correspondence between conformal structures and Beltrami differentials.

Fixing the complex structure $\mathcal{A}$ on $X$, we can associate a conformal structure to $X$ whose Beltrami differential is identically zero in every local coordinate belonging to $\mathcal{A}$. This conformal structure is called the **standard conformal structure** of $X$. We will always denote this structure by $\sigma_X$. A typical metric in $\sigma_X$ would then look like $\gamma(z)|dz|^2$ in every local coordinate $z$ in $\mathcal{A}$.

We have just seen that a conformal structure on a Riemann surface $X$ can be described in a one-to-one fashion by a Beltrami differential on $X$. Here is a natural question: How could one give a geometric description of a conformal structure on $X$? Answer: Just take a look at the "field of ellipses" of vectors of constant length on each tangent plane. Let us elaborate this in more details.

Consider a conformal structure $\sigma$ on $X$ whose Beltrami differential is $\mu$. For almost every $z$ in $X$, we can form concentric ellipses $||\mathbf{v}||$ =constant for $\mathbf{v}$ in $T_zX$, the tangent plane to $X$ at $z$ (see figure 1).

Note that these concentric ellipses depend only on $\sigma$ since any Riemannian metric in this conformal structure is a real multiple of any other when restricted to $T_zX$. It is easy to understand the geometry of these ellipses in terms of $\mu$. Consider a tangent vector $\mathbf{v} = u\partial/\partial x + v\partial/\partial y$ in $T_zX$. Passing to complex-variable notation, the condition $||\mathbf{v}|| = c$ means $|(u+iv) + \mu(z)(u-iv)| = c$ (cf. Appendix A). If $u + iv = re^{i\theta}$, this reduces to $|1 + |\mu(z)|e^{i(\arg\mu(z)-2\theta)}| = c/r$. So the

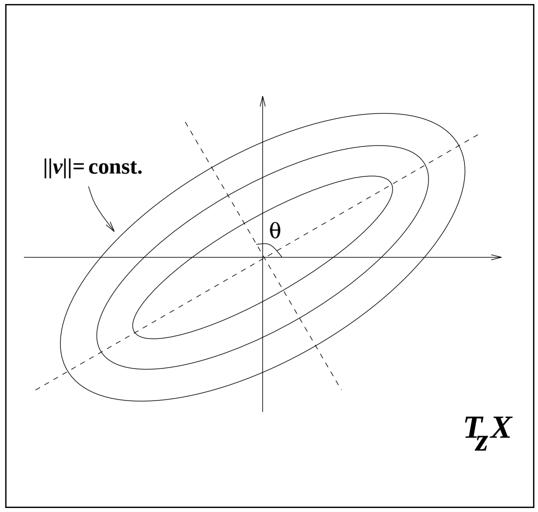


FIGURE 1

ratio of the major axis to the minor axis of this ellipse, measured in the standard conformal structure $\sigma_X$, is

$$K(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{3}$$

and the angle of elevation of the minor axis with respect to the horizontal direction in this particular local coordinate is

$$\theta(z) = \frac{1}{2} \arg \mu(z). \tag{4}$$

From (3) it follows in particular that $|\mu(z)|$ is a well-defined measurable function on $X$ (as can also be seen from the transformation rule (2)). It also shows that we get "circles" if we consider the ellipses coming from $\sigma_X$.

Conversely, given any **measurable field of ellipses** on $X$, i.e., a family of concentric ellipses on almost every tangent plane varying in a measurable way, we can find a conformal structure by going backward: Measure the ratio of the major axis to the minor axis in the standard conformal structure to obtain $K$ and measure the angle $\theta$ in a particular local coordinate $z$. Then from (3) and (4) find $\mu(z)$. The conformal structure will then be the conformal class of $|dz + \mu(z) d\bar{z}|^2$.

As a result, conformal structures on $X$ and measurable fields of ellipses on the tangent planes of $X$ are in fact the same objects.

A measurable field of ellipses is said to have **bounded dilatation** with respect to $\sigma_X$ if $||K||_\infty < +\infty$, where $||.||_\infty$ denotes the essential supremum over $X$. This means that the ellipses cannot be distorted arbitrarily large. We are allowed to make an error in drawing our 'circles' only within a certain bounded range. From (3) it is natural to define a conformal structure of bounded dilatation with respect to $\sigma_X$ as one whose Beltrami differential satisfies

$$||\mu||_\infty < 1.$$

As an example, the conformal class of

$$|dz + \frac{z^2}{|z|} d\bar{z}|^2 \tag{5}$$

does *not* have bounded dilatation on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ (see figure 2).

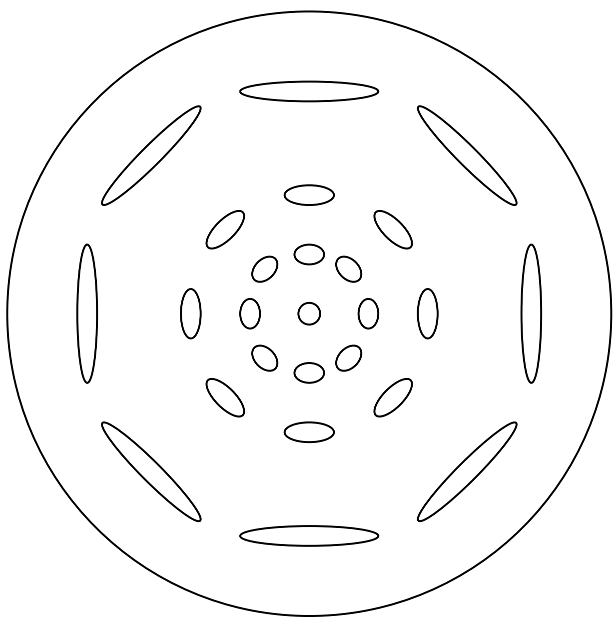

FIGURE 2

Yet another way of describing a conformal structure on a Riemann surface is through an algebraic approach. Fix a conformal structure $\sigma$ on $X$ and consider its field of ellipses on almost every tangent plane $T_z X$. We can define an $\mathbb{R}$-linear isomorphism $J_\sigma$ on $T_z X$ such that $J_\sigma^2 = -$identity. To this end, pick an orthogonal pair $(e_1, e_2)$ as in the figure and let $J_\sigma(e_1) = e_2$ and $J_\sigma(e_2) = -e_1$, and extend $J_\sigma$ by linearity. It is clear that $J_\sigma$ constructed in this way does not depend on the choice of the ellipse where $(e_1, e_2)$ was picked up. Such family of linear maps is an example of a measurable almost-complex structure. Strictly speaking, a **measurable almost-complex structure** on $X$ is a rule $z \mapsto J(z)$ which assigns to almost every $z$ on $X$ an $\mathbb{R}$-linear isomorphism $J(z)$ on $T_zX$ in a measurable fashion such that $J(z)^2 = -$identity.

For example, the almost-complex structure induced by $\sigma_X$ sends $\partial/\partial x$ to $\partial/\partial y$ and $\partial/\partial y$ to $-\partial/\partial x$ in every local coordinate $z = x + iy$. In the complex-variable notation, therefore, it sends $\partial/\partial z$ to $i\partial/\partial z$ (cf. Appendix A). For this reason, we usually regard it as the "multiplication by $i$" or the "rotation by 90°" in the positive direction on $T_zX$.

Conversely, consider a measurable almost-complex structure $z \mapsto J(z)$ such that the orientation of the pair $(\partial/\partial x, J(\partial/\partial x))$ is the same as $(\partial/\partial x, \partial/\partial y)$ for almost every $z$ on $X$. Fix a typical $z$ and pick an $\mathbb{R}$-linear isomorphism $\phi$ on $T_zX$ such that $\phi^{-1}J(z)\phi(\partial/\partial x) = \partial/\partial y$ and $\phi^{-1}J(z)\phi(\partial/\partial y) = -\partial/\partial x$. In other words, $\phi$ conjugates the action of $J$ with the multiplication by $i$. Consider concentric circles on $T_zX$ coming from $\sigma_X$ and take their image under $\phi$. The resulting concentric ellipses are well-defined since they depend neither on the local coordinate $z$ nor on the choice of $\phi$. Each ellipse is preserved under $J(z)$. In this way we obtain a field of ellipses, hence a conformal structure, out of an almost-complex structure (see figure 3).

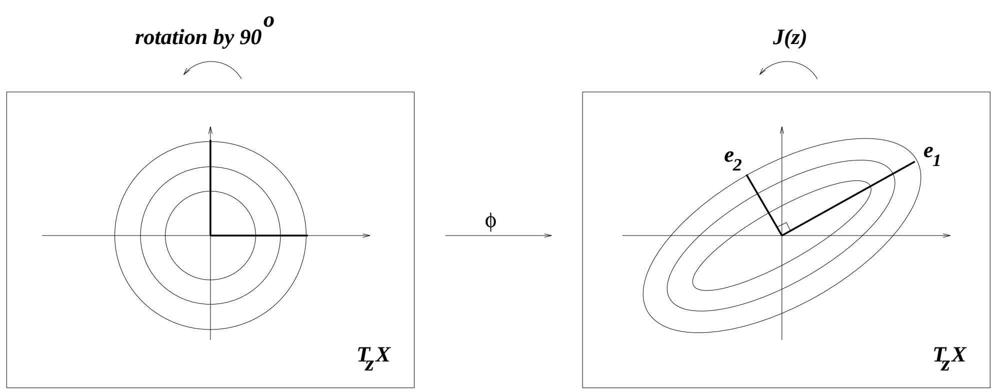


FIGURE 3

Now let us define an operation on conformal structures which corresponds to pulling back the Riemannian metrics in differential geometry. Roughly speaking, we can pull back a conformal structure by any smooth diffeomorphism, but because of the measurable nature of our objects, we can relax the condition of being smooth. For a moment, let $X$ and $Y$ be two Riemann surfaces and let $f : X \to Y$ be an orientation-preserving diffeomorphism. Given any conformal structure $\sigma$ on $Y$ we can pull it back to get a conformal structure $f^*\sigma$ on $X$. In fact, if $\sigma$ is the conformal class of $|dw + \mu(w)d\bar{w}|^2$ and if $w = f(z)$ is the local expression of $f$, then $f^*\sigma$ will be the conformal class of

$$|dz + \frac{f_{\bar{z}} + \mu(f(z))\bar{f}_{\bar{z}}}{f_z + \mu(f(z))\bar{f}_z} d\bar{z}|^2$$

where $f_z$ and $f_{\bar{z}}$ are complex partial derivatives of $f$ in local coordinate $z$ (cf. Appendix A). In particular, if $\sigma_Y$ is the standard conformal structure of $Y$, then

$$f^*\sigma_Y = \text{conformal class of } |dz + \frac{f_{\bar{z}}}{f_z} d\bar{z}|^2. \tag{6}$$

The Cauchy-Riemann equations will then show that $f : X \to Y$ is a conformal map iff $f^*\sigma_Y = \sigma_X$. In the language of almost-complex structures, this means that the derivative $Df(z)$ as an $\mathbb{R}$-linear map $T_zX \to T_{f(z)}Y$ commutes with the corresponding multiplications by $i$:

$$\begin{array}{ccc} T_zX & \xrightarrow{Df(z)} & T_{f(z)}Y \\ J \downarrow & & \downarrow J' \\ T_zX & \xrightarrow{Df(z)} & T_{f(z)}Y \end{array}$$

Now, all conformal structures are only measurable and it seems rather awkward to use smooth maps to pull them back. In fact, the same kind of operations as above can be defined even if $f$ is not a diffeomorphism. But in order to extend the class of maps for which the pull-back of conformal structures makes sense, we have to pick up those homeomorphisms for which the partial derivatives exist in some reasonable sense. It turns out that such homeomorphisms do exist: They are the so-called quasiconformal homeomorphisms.

Strictly speaking, a homeomorphism $f : X \to Y$ is called **quasiconformal** if at almost every point it has locally square-integrable generalized partial derivatives $f_z$ and $f_{\bar{z}}$ in every local coordinate $z$ (cf.

Appendix B) and

$$||\frac{f_{\bar{z}}}{f_z}||_\infty < 1. \tag{7}$$

Then we can imitate the case of a diffeomorphism to pull a conformal structure back by $f$, so the formula (6) still holds. As a result, $f : X \to Y$ is quasiconformal iff $f^*\sigma_Y$ has bounded dilatation with respect to $\sigma_X$.
What is more exciting is the fact that even for a quasiconformal homeomorphism $f : X \to Y$, $f^*\sigma_Y = \sigma_X$ implies $f$ is conformal. In other words, a quasiconformal homeomorphism $f$ with $f_{\bar{z}} = 0$ is actually holomorphic, which is a generalization of Cauchy-Riemann equations for homeomorphisms. This fact is known as *Weyl's Lemma*. The quantity on the left side of (7) is called the **maximal dilatation** of $f$. By Weyl's lemma, $f$ is conformal iff its maximal dilatation is zero.

Quasiconformal homeomorphisms have extraordinary analytic properties. They can also be characterized by simple geometric properties for which we refer the reader to [**2**] (see also Appendix B, Theroem B-1).

Now, let us stop for a moment and look at what we have already constructed. From a complex structure on a surface we can construct a (standard) conformal structure, or equivalently a field of ellipses, or equivalently an almost-complex structure:

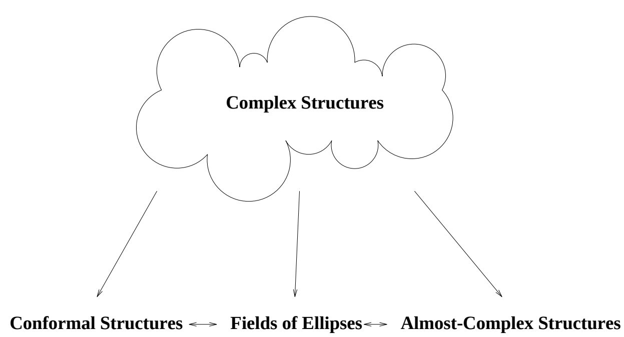


FIGURE 4

Here we come up with the most delicate question: Is it possible to draw an arrow from downstairs to upstairs in this diagram? In other words, is it true that given
(a) a conformal structure $\sigma$, or
(b) a measurable field of ellipses, or

(c) a measurable almost-complex structure $J$
on a surface $X$, one can find a complex structure $\mathcal{A}$ on $X$ such that
(a') $\sigma$ is the standard conformal structure of $(X, \mathcal{A})$? or
(b') the circles in the tangent planes of $(X, \mathcal{A})$ are the given ellipses? or
(c') the action of $J$ is the same as multiplication by $i$ in $(X, \mathcal{A})$?

## 3. THE THEOREM

The *Measurable Riemann Mapping Theorem* provides a complete answer to the above question. It turns out that the answer is affirmative provided that we impose a boundedness condition on the given conformal structure. Roughly speaking, the theorem says that the answer is positive if the conformal structure has a reasonable deviation from the standard conformal structure of *some* complex structure on the surface. In other words, if there is a complex structure on $X$ such that $\sigma$ has bounded dilatation with respect to $\sigma_X$, then we can re-define the complex structure of $X$ such that $\sigma$ is standard in the new $\mathbb{C}$-analytic atlas. In this case $\sigma$, or the field of ellipses, or the almost-complex structure, is called **integrable**.

Let us formulate this question in a more convenient language. Let $X$ be given a fixed complex structure and let $\sigma$ have bounded dilatation with respect to $\sigma_X$.

**Fact**(†) *To say that $\sigma$ is integrable means that there exists a Riemann surface $Y$ and a quasiconformal homeomorphism $f : X \to Y$ such that $f^*\sigma_Y = \sigma$.*

In fact, if $f$ is such a map, equip $X$ with the pull-back complex structure (whose charts are $f$ composed with the charts of $Y$). Then $\sigma$ is standard in this new complex structure. On the other hand, if $\sigma$ is standard in some new complex structure on $X$, call this new Riemann surface $Y$ and consider the identity map $\mathrm{id} : X \to Y$. Obviously, $(\mathrm{id})^*\sigma_Y = \sigma$.

Moreover, it is easy to prove the uniqueness of such $Y$ (if any). If $g : X \to Z$ is another quasiconformal homeomorphism with $g^*\sigma_Z = \sigma$, then $h = g \circ f^{-1} : Y \to Z$ satisfies $h^*\sigma_Z = \sigma_Y$, so it is conformal. Conversely, if $h : Y \to Z$ is a conformal homeomorphism, then $g = h \circ$

$f : X \to Z$ satisfies $g^*\sigma_Z = \sigma$. We conclude that *the solution* $f : X \to Y$ *is unique up to post-composition with a conformal homeomorphism of* $Y$.

As the basic case, let us assume that $X = \mathbb{D}$, the unit disk in the complex plane. Let $\sigma$ have bounded dilatation with respect to $\sigma_{\mathbb{D}}$, which means the corresponding Beltrami differential $\mu$ satisfies $||\mu||_\infty < 1$. Then by (6) a quasiconformal homeomorphism $f : \mathbb{D} \to Y$ satisfies $f^*\sigma_Y = \sigma$ iff it is a solution of the equation

$$\frac{f_{\bar{z}}}{f_z} = \mu, \qquad ||\mu||_\infty < 1$$

which is called the **Beltrami equation**.

Local solutions of this equation were found by Gauss for a real analytic $\mu$. Beltrami himself extensively used the equation in his works on surface theory [**4**]. C.B. Morrey was the first who proved the existence of global solutions of the Beltrami equation for measurable $\mu$. Nevertheless, his proof had been overlooked by Ahlfors and many others for almost 20 years since it was totally written in the language of partial differential equations. The early version of the proof for Hölder continuous $\mu$ was already published by Ahlfors when Bers realized the connection of Morry's work with quasiconformal mappings. In their 1960 fundamental paper, Ahlfors and Bers proved a much more powerful version which also gives analytic dependence on parameters. This seemingly technical improvement turned out to be of tremendous importance for the theory of Teichmüller spaces.

According to the uniformization theorem, the Riemann surface $Y$ above is conformally equivalent to either $\mathbb{D}$ or the complex plane $\mathbb{C}$. But the theorem shows that $Y$ can always be chosen to be $\mathbb{D}$, which means that the Beltrami equation has solutions $f$ which are quasiconformal homeomorphisms $\mathbb{D} \to \mathbb{D}$. This will show that $Y$ can never be $\mathbb{C}$, i.e., there is no quasiconformal homeomorphism between $\mathbb{D}$ and $\mathbb{C}$ (see Corollary 1 below).

**Theorem 1 (Ahlfors-Bers-Morrey).** *Let* $\sigma$ *be a measurable conformal structure of bounded dilatation on the unit disk* $\mathbb{D}$. *Then there exists a quasiconformal homeomorphism* $f : \mathbb{D} \to \mathbb{D}$ *with* $f^*\sigma_{\mathbb{D}} = \sigma$. $f$ *is unique up to post-composition with a conformal homeomorphism of* $\mathbb{D}$. *Moreover, if* $\{\sigma^t\}$ *is a family of such conformal structures which depends continuously, smoothly, or analytically on a parameter* $t$, *then* *there exists a family* $\{f^t\}$ *of quasiconformal homeomorphisms of* $\mathbb{D}$ *with* $(f^t)^*\sigma_{\mathbb{D}} = \sigma^t$ *which depends on* $t$ *in the same way as* $\{\sigma^t\}$ *does.*

As a simple consequence of the theorem, we have the following

**Corollary 1.** *There is no quasiconformal homeomorphism between* $\mathbb{D}$ *and* $\mathbb{C}$.

In fact, let $g : \mathbb{D} \to \mathbb{C}$ be such a homeomorphism. Then $g^*\sigma_{\mathbb{C}}$ has bounded dilatation with respect to $\sigma_{\mathbb{D}}$. By the theorem above, there is $f : \mathbb{D} \to \mathbb{D}$ such that $f^*\sigma_{\mathbb{D}} = g^*\sigma_{\mathbb{C}}$, or $(f \circ g^{-1})^*\sigma_{\mathbb{D}} = \sigma_{\mathbb{C}}$. But this means that $f \circ g^{-1} : \mathbb{C} \to \mathbb{D}$ is conformal, hence constant by Liouville's theorem.

It might seem that there are genuine global obstructions to integrability for a general Riemann surface. But that is just not true. In fact the general case is a consequence of the 'local' case $X = \mathbb{D}$.

Let $X$ be an arbitrary Riemann surface with a $\mathbb{C}$-analytic atlas $\mathcal{A}$. Let $\sigma$ be a measurable conformal structure of bounded dilatation with respect to $\sigma_X$. Cover $X$ by a countable union of charts $(U, z)$ in $\mathcal{A}$, where $z : U \to \mathbb{D}$ is a homeomorphism. In each such local coordinate, $\sigma$ is the conformal class of $|dz + \mu(z)d\bar{z}|^2$ with $z$ in $\mathbb{D}$ and $||\mu||_\infty < 1$. By the above theorem, there is a quasiconformal homeomorphism $f : \mathbb{D} \to \mathbb{D}$ such that $f^*\sigma_{\mathbb{D}} = \sigma$. Choose $(U, f \circ z)$ as a chart in a new atlas $\mathcal{B}$. If $(V, w)$ is also in $\mathcal{A}$ with $U \cap V \neq \emptyset$ and if $(V, g \circ w)$ is the corresponding new chart in $\mathcal{B}$, then the change of coordinate $(g \circ w) \circ (f \circ z)^{-1}$ is conformal since it preserves $\sigma_{\mathbb{D}}$:

$$\begin{aligned}[(g \circ w) \circ (f \circ z)^{-1}]^*\sigma_{\mathbb{D}} &= (f^{-1})^*(w \circ z^{-1})^*g^*\sigma_{\mathbb{D}} \\ &= (f^{-1})^*(w \circ z^{-1})^*\sigma \\ &= (f^{-1})^*\sigma \\ &= \sigma_{\mathbb{D}}.\end{aligned}$$

Therefore $\mathcal{B}$ is actually a $\mathbb{C}$-analytic atlas on $X$ with respect to which $\sigma$ is standard. By fact (†) above, we have proved

**Theorem 2.** *Let* $X$ *be a Riemann surface and* $\sigma$ *be a measurable conformal structure of bounded dilatation with respect to* $\sigma_X$. *Then there exists a Riemann surface* $Y$ *and a quasiconformal homeomorphism* $f : X \to Y$ *such that* $f^*\sigma_Y = \sigma$. $f$ *is unique up to post-composition with a conformal homeomorphism of* $Y$.

When $X$ is the Riemann sphere $\bar{\mathbb{C}}$, it follows from the uniformization theorem that every Riemann surface $Y$ homeomorphic to $X$ is in fact conformally equivalent to $X$. Moreover, every conformal homeomorphism of the sphere is a complex Möbius transformation which is uniquely determined by its image at 3 distinct points. Consequently, one has

**Theorem 3.** *Let $\sigma$ be a measurable conformal structure of bounded dilatation on the Riemann sphere $\bar{\mathbb{C}}$. Then there exists a unique quasiconformal homeomorphism $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ with $f(0) = 0,\ f(1) = 1,\ f(\infty) = \infty$ such that $f^*\sigma_{\bar{\mathbb{C}}} = \sigma$.*

The proof of Theorem (2) also reveals the following fact. If we consider a conformal class of *continuous* Riemannian metrics on a Riemann surface, we do not have to impose any boundedness condition on it. In fact, given any such conformal class on $X$, we equip $X$ with an arbitrary complex structure and we take a covering of $X$ by a bunch of disks compactly contained in a given covering. It follows that the function $|\mu|$ is uniformly less than 1 on each disk. Then we repeat the same argument as before.

**Corollary 2.** *Every continuous conformal structure on a smooth surface is integrable.*

As an example, the conformal structure (5) on $\mathbb{D}$ does not have bounded dilatation with respect to $\sigma_{\mathbb{D}}$. Nevertheless, it is continuous and so integrable. It can be checked that the Riemann surface on which (5) is standard is conformally equivalent to $\mathbb{C}$. In fact, the function $f(z) = z(1-|z|)^{-2}$ defines the required conformal homeomorphism $\mathbb{D} \to \mathbb{C}$.

## 4. An Application: Polynomial-Like Maps

There have been several results in conformal dynamics in recent years whose proofs use the measurable Riemann mapping theorem in an essential way. These include Sullivan's proof of the no wandering domain conjecture [**18**], Shishikura's sharp estimates on the number of non-repelling cycles of a rational map on the sphere [**16**], construction of Herman rings by quasiconformal surgery [**16**], and the theory of polynomial-like maps of Douady and Hubbard [**7**] which is an important tool for studying the quadratic family $\{z \mapsto z^2 + c\}$ and the Mandelbrot set. Here we give a basic application of Theorem 3 in the theory of polynomial-like maps.

Let $P$ be a polynomial of degree $d > 1$ with complex coefficients. For large $|z|$, it behaves like $z \mapsto cz^d$. Therefore, for large $R > 0$, it maps the disk $U = \{z : |z| < R\}$ onto a larger topological disk $V$ with $V \supset \bar{U}$ and $P : U \to V$ is a $d$-to-1 proper holomorphic map. We define the **filled Julia set** $K(P)$ as the set of all points $x$ whose orbit $\{P^n(x)\}_{n \geq 0}$ remains bounded in $\mathbb{C}$:

$$K(P) = \bigcap_{n \geq 0} P^{-n}(\bar{U}).$$

This is a compact subset of $U$ which is usually wild, fractal-shaped.

The behavior of polynomials for large $|z|$ suggests that we study the following situation: Let $U$ and $V$ be open topological disks with smooth boundaries such that $\bar{U} \subset V$. Let $f : U \to V$ be a proper holomorphic map of degree $d > 1$. We call $f$ a **polynomial-like** map. Imitating the polynomial case, we define the filled Julia set $K(f)$ as the set of all $x$ in $U$ whose orbit under $f$ never leaves $U$. Specifically, $K(f) = \cap_{n \geq 0} f^{-n}(\bar{U})$.

We want to show that a polynomial-like map has the same iterative behavior as an actual polynomial near the corresponding filled Julia sets. In fact we show that there is a *quasiconformal conjugacy* between any polynomial-like map and an actual polynomial which is conformal in the interior of the filled Julia sets. In this case, we say that the two maps are **hybrid equivalent**.

**Theorem 4 (Douady-Hubbard).** *Let $f : U \to V$ be a polynomial-like map of degree $d > 1$. Then there exists a polynomial $P$ of degree $d$ and a quasiconformal homeomorphism $\phi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ with $\phi(\infty) = \infty$ which conjugates $f$ and $P$ in a neighborhood of their filled Julia sets:*

$$\phi(f(z)) = P(\phi(z)).$$

*Moreover, $\phi_{\bar{z}} = 0$ on $K(f)$, so the conjugacy is conformal in the interior of $K(f)$.*

The key point is to extend the action of $f$ to the whole sphere by *gluing* it to the polynomial $z \mapsto z^d$. Then we define an invariant conformal structure on the sphere and use Theorem 3 to show that it is integrable. To this end, let $h : \bar{\mathbb{C}} \backslash V \to \{z : |z| \geq 2\}$ be a conformal homeomorphism with $h(\infty) = \infty$, and extend it in a smooth way to a diffeomorphism $h : \bar{\mathbb{C}} \backslash U \to \{z : |z| \geq 2^{1/d}\}$ such that

$$h(f(z)) = (h(z))^d \qquad z \in \partial U.$$

Now the smooth function

$$\tilde{f}(z) = \begin{cases} f(z) & z \in U \\ h^{-1} \circ (h(z))^d & z \in \bar{\mathbb{C}} \backslash U \end{cases}$$

is the desired extension of $f$ which is conformally conjugate to $z \mapsto z^d$ outside of $V$. Define a measurable conformal structure $\sigma$ on the sphere as follows. Let $\sigma$ be $\sigma_{\bar{\mathbb{C}}}$ on $\bar{\mathbb{C}} \backslash \bar{V}$. We can pull it back from $\bar{\mathbb{C}} \backslash \bar{V}$ to $V \backslash \bar{U}$ by $\tilde{f}$ to define $\sigma$ there. Next, we define $\sigma$ on $U \backslash f(\bar{U})$ as the pull-back of $\sigma$ on $V \backslash \bar{U}$ by $\tilde{f}$. This process of taking pull-backs can be continued to define $\sigma$ everywhere except $K(f)$. We simply put $\sigma = \sigma_{\bar{\mathbb{C}}}$ on $K(f)$. Note that $\sigma$ obtained in this way is invariant under $\tilde{f}$ and has bounded dilatation, since after the first step, all successive pull-backs are done by a holomorphic map which does not change the dilatation.

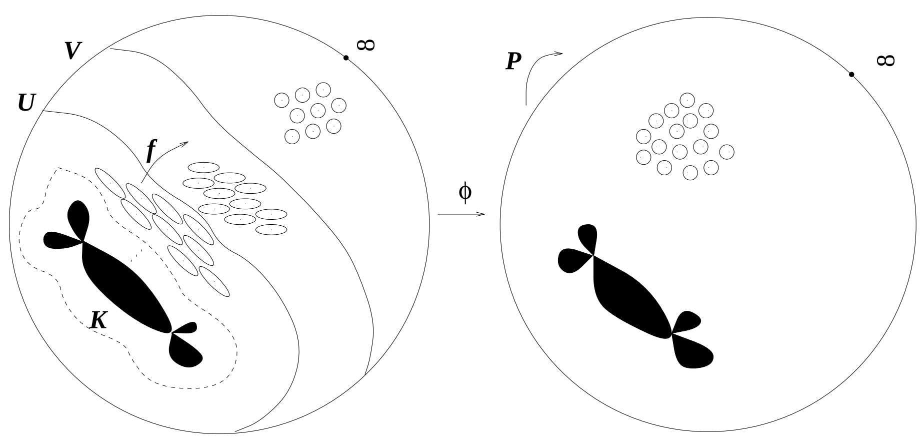


Figure 5

By Theorem 3 there is a quasiconformal homeomorphism $\phi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ fixing $\infty$ such that $\phi^* \sigma_{\bar{\mathbb{C}}} = \sigma$. Since $\sigma$ is invariant under $\tilde{f}$, $\sigma_{\bar{\mathbb{C}}}$ will be invariant under $P = \phi \circ \tilde{f} \circ \phi^{-1}$, so $P$ is holomorphic. Also $P$ is a $d$-to-1 proper map fixing $\infty$. Therefore $P$ is a polynomial of degree $d$.

Note that $\phi_{\bar{z}} = 0$ on $K(f)$ since $\sigma$ was defined to be standard there.

## 5. Further Readings

This paper in only an introduction to the very basic features of the measurable Riemann mapping theorem. Here we briefly mention a few references which cover most of the topics we touched in this paper.

Gauss's proof of the existence of isothermal coordinates can be found in his collected works [**9**]. There is an English translation of the main part of his paper in [**17**]. There is a nice proof due to Douady and Fathi of the existence of solutions of the Beltrami equation (without dependence on parameters) in [**6**]. They first prove the result for an analytic $\mu$ using a complex-time ordinary differential equation and then they apply standard approximation techniques to find the solution in the general case. The general problem of integrability of smooth almost-complex structures is answered by a fundamental theorem of Newlander-Nirenberg [**15**] which gives a simple proof in the smooth two-dimensional case.

The most comprehensive study of quasiconformal maps can be found in [**12**]. For an excellent short introduction, including a proof of Theorem 1 in this paper and basic applications to Teichmüller theory, see [**2**]. Another topic of main interest is the theory of quasiconformal maps in higher dimensions [**10**].

Teichmüller theory is the study of the space of quasiconformal deformations of a Riemann surface. It has emerged naturally as an attempt to answer Riemann's moduli problem for compact surfaces. Good introductions to this subject with applications to quadratic differentials and Fuchsian groups can be found in [**1**], [**8**], and [**11**].

A major area in which quasiconformal maps have been successfully applied is conformal dynamics. For an excellent introduction to classical Fatou-Julia theory, see [**13**]. Standard applications of quasiconformal maps in conformal dynamics include [**7**], [**16**], and [**18**]. For a general exposition of how to use quasiconformal maps in dynamics (and some other areas) see [**19**]. Techniques of Teichmüller theory have nice applications in renormalization theory, for which we refer to the last chapter of [**5**].

## Appendix A. Complex-Variable Notations on a Surface

Let $X$ be a $C^\infty$ smooth, connected, oriented surface. $TX$, the real rank 2 tangent bundle of $X$, and its dual $T^*X$, the real rank 2 cotangent bundle of $X$, can be complexified by taking the tensor products $T_{\mathbb{C}}X = TX \otimes \mathbb{C}$ and $T^*_{\mathbb{C}}X = T^*X \otimes \mathbb{C}$. Therefore, in a local coordinate $(x, y)$ on $X$, a typical section of $T_{\mathbb{C}}X$ and $T^*_{\mathbb{C}}X$ can be written as

$$u(x,y)\frac{\partial}{\partial x} + v(x,y)\frac{\partial}{\partial y},$$

and

$$u(x,y)dx + v(x,y)dy$$

respectively, where $u$ and $v$ are smooth, complex-valued functions of $x$ and $y$. We define two special sections

$$dz = dx + idy \qquad \text{and} \qquad d\bar{z} = dx - idy,$$

which form a basis for $T^*_{\mathbb{C}}X$ in every local coordinate $(x, y)$. We use notations $\partial/\partial z$ and $\partial/\partial \bar{z}$ for the dual basis in the same local coordinate:

$$dz(\frac{\partial}{\partial z}) = 1, \qquad dz(\frac{\partial}{\partial \bar{z}}) = 0,$$

$$d\bar{z}(\frac{\partial}{\partial z}) = 0, \qquad d\bar{z}(\frac{\partial}{\partial \bar{z}}) = 1.$$

Therefore

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}).$$

We may regard $\partial/\partial z$ and $\partial/\partial \bar{z}$ as differential operators acting on smooth functions $f : U \to \mathbb{C}$, where $U$ is the domain on which the local coordinate $(x, y)$ is defined. The action is simply defined by

$$\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}).$$

We usually use indices for partial derivatives, like $f_z$ for $\partial f/\partial z$, etc. Now the differential $df$ of a function $f$ satisfies

$$df = f_x dx + f_y dy = f_z dz + f_{\bar{z}} d\bar{z}.$$

When $X$ has a complex structure which makes it into a Riemann surface, the complex one-dimensional subspace generated by $\partial/\partial z$ in every local coordinate $z = x + iy$ defines a sub-bundle of $T_{\mathbb{C}}X$ which we denote by $T_{hol}X$. This follows at once from the fact that the change of coordinates are holomorphic. $T_{hol}X$ is isomorphic to $TX$ as a real rank

2 bundle via $\partial/\partial z \mapsto \partial/\partial x$ and $i\partial/\partial z \mapsto \partial/\partial y$. It follows from Cauchy-Riemann equations that a function $f$ is holomorphic iff in every local coordinate $z = x + iy$, $f_{\bar{z}} = 0$, or $df$ belongs to the cotangent bundle $T^*_{hol}X$.

Back to the case of a smooth surface. A measurable Riemannian metric on $X$ can be locally written as

$$g = Edx^2 + 2Fdxdy + Gdy^2, \tag{8}$$

where $E, F$, and $G$ are (Lebesgue) measurable functions of $x, y$ such that $E > 0$, $G > 0$, and $EG - F^2 > 0$ almost everywhere. The length of a tangent vector

$$\mathbf{v} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

is given by

$$||\mathbf{v}||^2 = Eu^2 + 2Fuv + Gv^2.$$

If we use complex notation, the vector $\mathbf{v}$ has a representation

$$\mathbf{v} = (u + iv)\frac{\partial}{\partial z} + (u - iv)\frac{\partial}{\partial \bar{z}}. \tag{9}$$

A direct calculation shows that $g$ in (8) can be expressed as

$$g(z) = \gamma(z)|dz + \mu(z)d\bar{z}|^2, \tag{10}$$

where $z = x + iy$ and

$$\gamma(z) = \frac{1}{2}\left(\frac{E+G}{2} + \sqrt{EG - F^2}\right),$$

$$\mu(z) = \frac{1}{\gamma(z)}\left(\frac{E-G}{4} + i\frac{F}{2}\right).$$

It is easy to check that $\gamma > 0$ and $|\mu| < 1$ almost everywhere. The expression in (10) means that the length of a tangent vector $\mathbf{v}$ in (9) satisfies

$$||\mathbf{v}||^2 = \gamma|(u + iv) + \mu(u - iv)|^2.$$

The metric $g$ is called **conformal** in the local coordinate $z$ if $\mu(z) \equiv 0$, so that it takes the form $\gamma(z)|dz|^2$. This notion is not well-defined in the case of a surface for which the change of coordinates are just smooth diffeomorphisms. In fact, if $z \mapsto w$ is such a change of coordinates, the metric in $w$ will take the form

$$g(w) = \gamma(w)|z_w dw + z_{\bar{w}} d\bar{w}|^2 = \gamma(w)|z_w|^2|dw + \frac{z_{\bar{w}}}{z_w}d\bar{w}|^2,$$

which is *not* conformal in $w$. However, if $X$ is equipped with a complex structure which makes it into a Riemann surface, the change of coordinates are holomorphic and so $z_{\bar{w}} = 0$, which shows that the metric will be conformal in $w$, too. Therefore, on a Riemann surface the notion of a conformal metric does not depend on the particular choice of the local coordinates.

## APPENDIX B. QUASICONFORMAL HOMEOMORPHISMS

Here we give equivalent definitions for quasiconformality in the planar case. These 'local' definitions can then be used to define the concept of quasiconformality on Riemann surfaces (cf. [**2**],[**12**]).

In what follows we always assume that $f : U \to V$ is an orientation-preserving homeomorphism between two regions in the plane.

The homeomorphism $f$ is said to have **generalized partial derivatives** if there exist integrable functions $\xi$ and $\eta$ such that

$$\int_U \xi h \; dxdy = -\int_U f h_z \; dxdy \quad \text{and} \quad \int_U \eta h \; dxdy = -\int_U f h_{\bar{z}} \; dxdy,$$

for every smooth function $h : U \to \mathbb{C}$ with compact support. In this case we write $\xi = f_z$ and $\eta = f_{\bar{z}}$ in the generalized sense.

We say that $f$ has **locally square-integrable generalized partial derivatives**, and we write $f \in W^1_{loc}$, if it has generalized partial derivatives $f_z$ and $f_{\bar{z}}$ in $U$ and for every compact set $E \subset U$ we have

$$\int_E |f_z|^2 \; dxdy < \infty \quad \text{and} \quad \int_E |f_{\bar{z}}|^2 \; dxdy < \infty.$$

$f$ is called **absolutely continuous on lines** if its restriction to almost every horizontal and vertical line in $U$ is absolutely continuous. It follows from classical real analysis that the partial derivatives $f_x$ and $f_y$, and so $f_z$ and $f_{\bar{z}}$, exist almost everywhere in $U$.

An **annulus** is a subset of the plane homeomorphic to a 'round' annulus $A(1,r) = \{z : 1 < |z| < r\}$. We set $A(1,\infty) = \{z : |z| > 1\}$. It follows from the uniformization theorem that any annulus $A$ is conformally homeomorphic to a *unique* $A(1,r)$, $1 < r \leq \infty$. We then define the **modulus** of $A$ by

$$\operatorname{mod}(A) = \frac{1}{2\pi} \log r,$$

where $\log\infty = \infty$. It follows that the modulus is a conformal invariant: Two annuli $A$ and $A'$ are conformally homeomorphic iff $\mathrm{mod}(A)$=$\mathrm{mod}(A')$.

The following theorem can be used as the definition of quasiconformality.

**Theorem B-1.** *Let $K \geq 1$. For an orientation-preserving homeomorphism $f : U \to V$ the following conditions are equivalent:*

(i) *$f \in W^1_{loc}$ and $|\dfrac{f_{\bar z}}{f_z}| \leq \dfrac{K-1}{K+1}$ almost everywhere in $U$,*

(ii) *$f$ is absolutely continuous on lines and $|\dfrac{f_{\bar z}}{f_z}| \leq \dfrac{K-1}{K+1}$ almost everywhere in $U$,*

(iii) *For every annulus $A \subset U$, $K^{-1}\mathrm{mod}(A) \leq \mathrm{mod}(f(A)) \leq K\mathrm{mod}(A)$,*

(iv) *For almost every $x$ in $U$, $\limsup_{r\to 0} \dfrac{\max\{|f(x)-f(y)| : |x-y| = r\}}{\min\ \{|f(x)-f(y)| : |x-y| = r\}} \leq K$.*

The homeomorphism $f$ is called $K$-**quasiconformal** if it satisfies any (hence all) of the above conditions.

**Definition.** Let $f : X \to Y$ be an orientation-preserving homeomorphism between Riemann surfaces. $f$ is called $K$-**quasiconformal** if its local representations in charts of $X$ and $Y$ are all $K$-quasiconformal homeomorphisms of planar regions. It is called **quasiconformal** if it is $K$-quasiconformal for some $K \geq 1$.

It is easy to see that this definition coincides with one we gave earlier in terms of the pull-back of conformal structures.

The following theorem summarizes the basic properties of quasiconformal homeomorphisms.

**Theorem B-2.**
(a) *If $f : X \to Y$ is $K_1$-quasiconformal and $g : Y \to Z$ is $K_2$-quasiconformal, then $g \circ f : X \to Z$ is $K_1K_2$-quasiconformal.*
(b) *$f : X \to Y$ is $K$-quasiconformal iff $f^{-1} : Y \to X$ is $K$-quasiconformal.*
(c) *$f : X \to Y$ is 1-quasiconformal iff $f$ is conformal.*

(d) *If $f$ is quasiconformal, then $f_z \neq 0$ almost everywhere.*
(e) *If $f$ is quasiconformal, then it maps sets of measure zero to sets of measure zero.*

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DEPARTMENT OF MATHEMATICS, SUNY AT STONY BROOK, NY 11794
*E-mail address*: zakeri@math.sunysb.edu

DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF NEW YORK, NY 10036
*E-mail address*: mzeinali@email.gc.cuny.edu